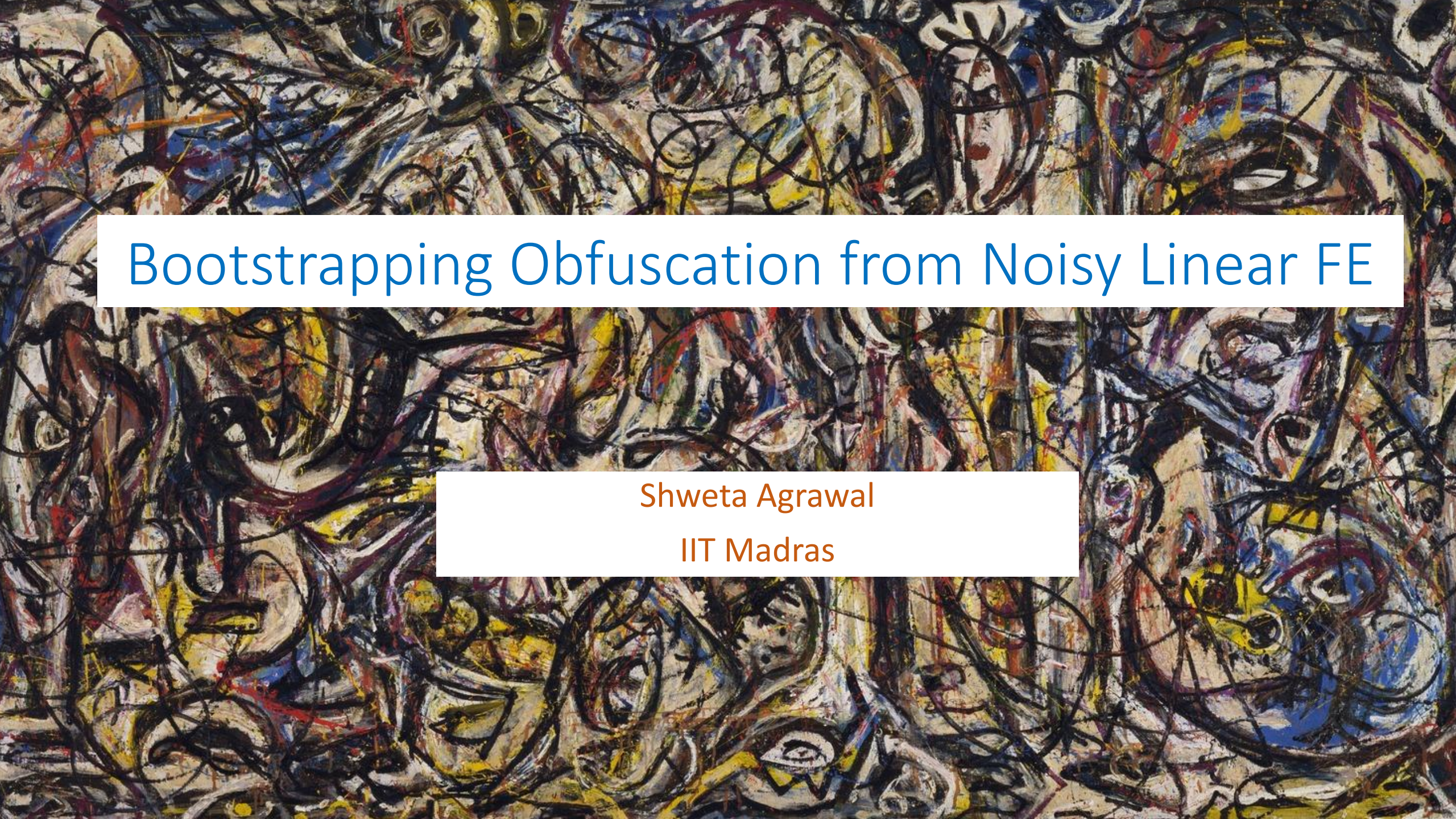

Bootstrapping Obfuscation from Noisy Linear FE

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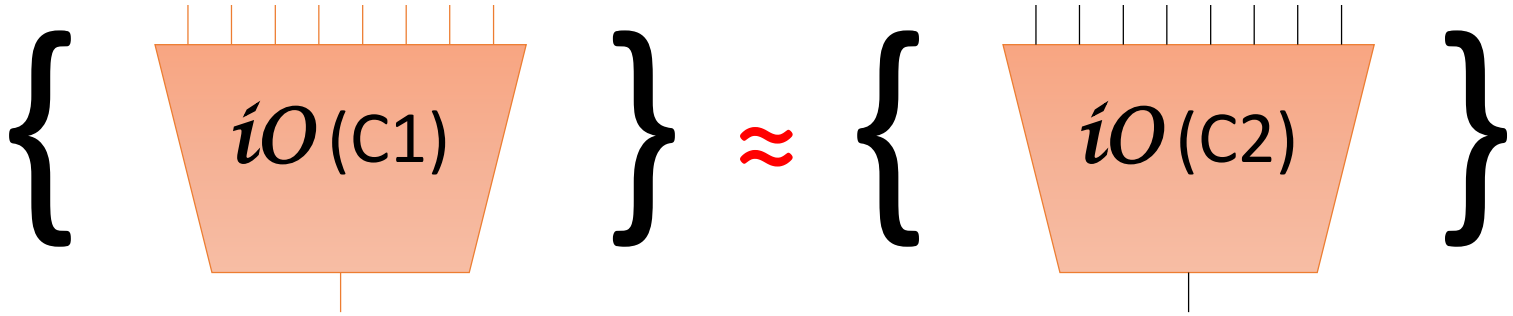
Indistinguishability Obfuscator $i\mathcal{O}$ [BGI+01]

"Which one of two equivalent circuits $C_1 \equiv C_2$ is obfuscated?"

$C_1 \equiv C_2$, meaning

- Same size $|C_1| = |C_2|$
- Same truth table $TB(C_1) = TB(C_2)$



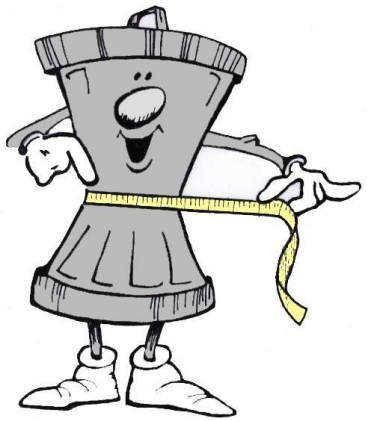
Quest: Finding an efficient compiler $i\mathcal{O}$

Does $i\mathcal{O}$ exist?

- Direct Constructions
  - All based on multilinear maps [GGH13,CLT13,GGH15]
  - Same template in all works (all eggs in same basket?)
  - Many attacks, fixes, repeat: hard to understand security

- Bootstrapping based constructions

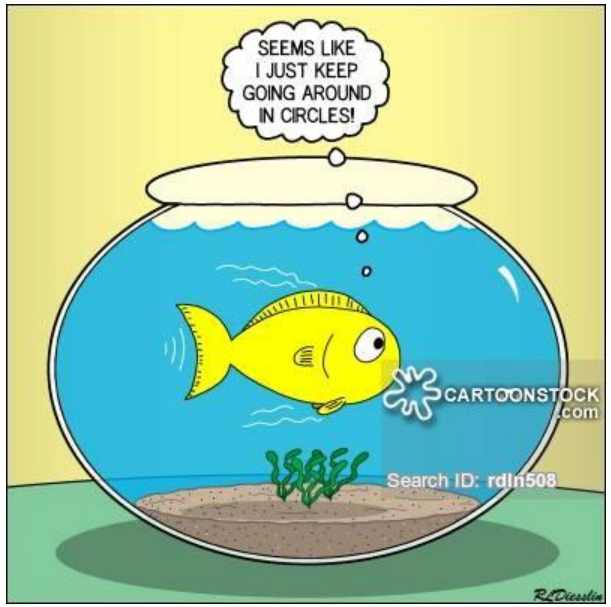
Bootsrapping Based Constructions: Reduce, Reduce, Reduce



- What is the minimum functionality needed for iO?
- How much can we “clean up” assumptions?
- Much progress
- State of art: degree 3 multilinear maps and degree 3 “block local” pseudorandom generators [LT17].
- Not clear how to instantiate deg 3 maps

Can we base $i\mathcal{O}$ on anything else?

- *Functional Encryption* supporting computation of degree ≥ 3 polynomials
- Should be good news except…..
  - All constructions of functional encryption themselves based on multilinear maps ☹

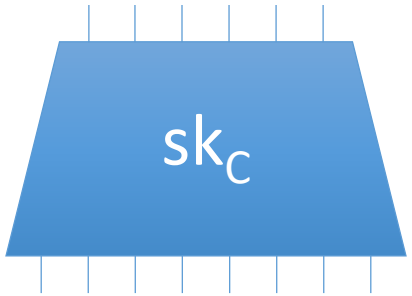


Functional Encryption

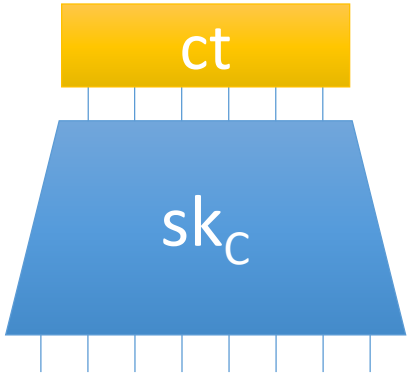
(mpk, msk) ← Setup(1^n)

Enc(mpk, m):

Kgen(msk, C):

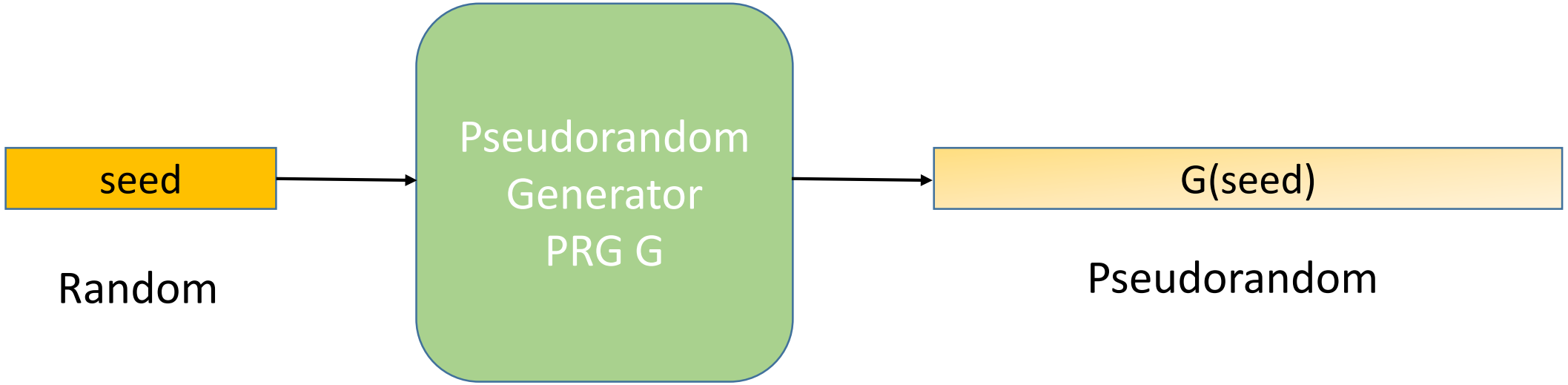


Dec(sk_C, ct):



The State of Affairs

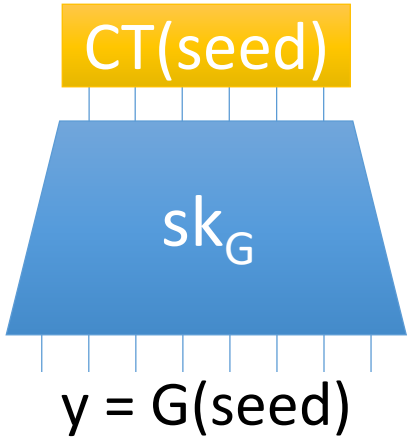
- Using bilinear maps, have FE for degree 2 polys
- For $i\mathcal{O}$, need FE for degree (at least) 3 polys
- Where does degree 3 come from?
- Arithmetic degree required to compute a PRG.



Expansion/stretch: Difference in output and input lengths, i.e |G(seed)| - |seed|

The State of Affairs

- Need FE to compute G(seed).



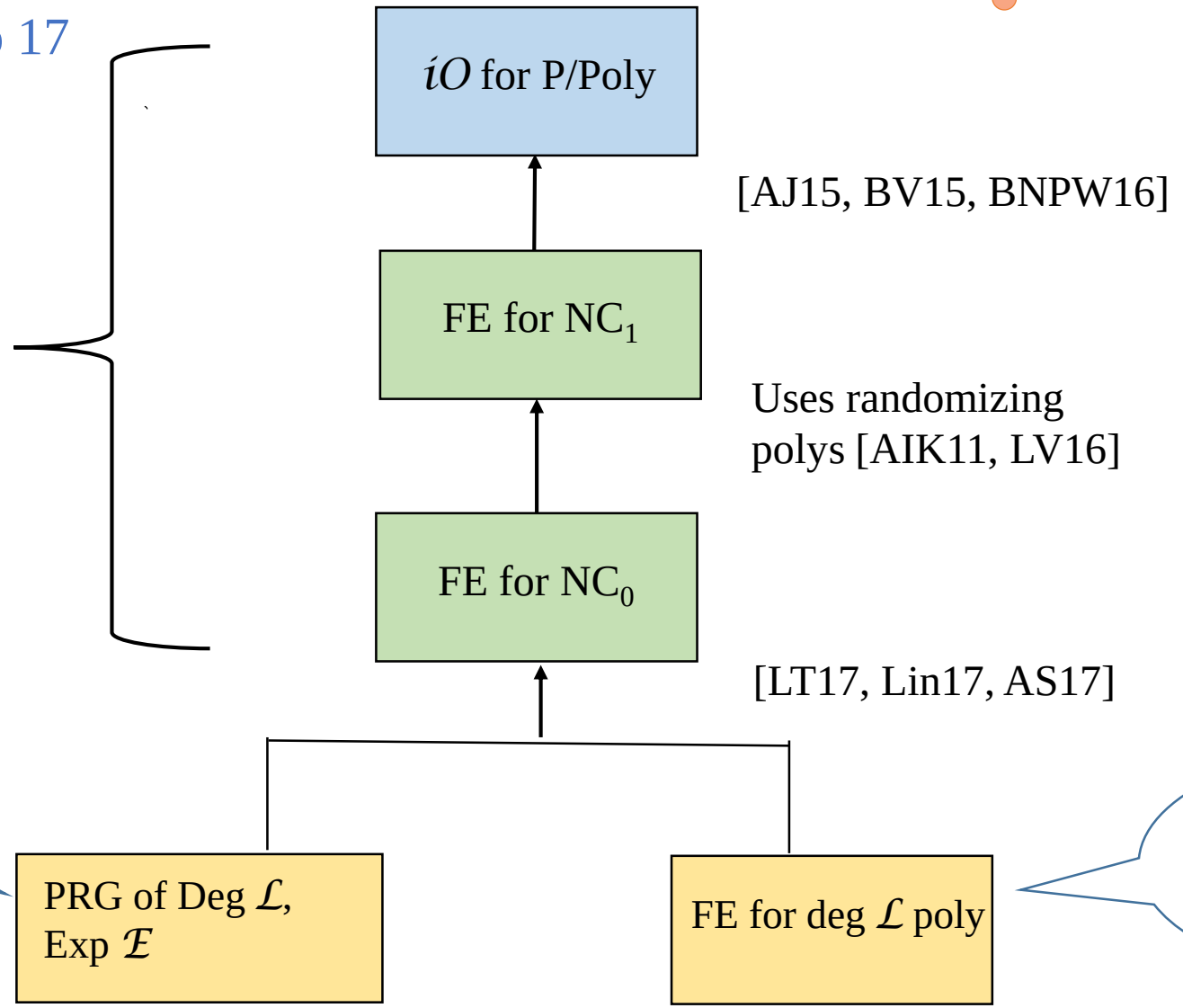
- Represent G as polynomial.
- G associated with pair $(\mathcal{L},\mathcal{E})$ where poly of degree $\mathcal{L}$ is required to compute PRG of expansion $\mathcal{E}$.

Previously….

Recall, G associated with $(\mathcal{L},\mathcal{F})$

Lin-Tessaro, Crypto 17

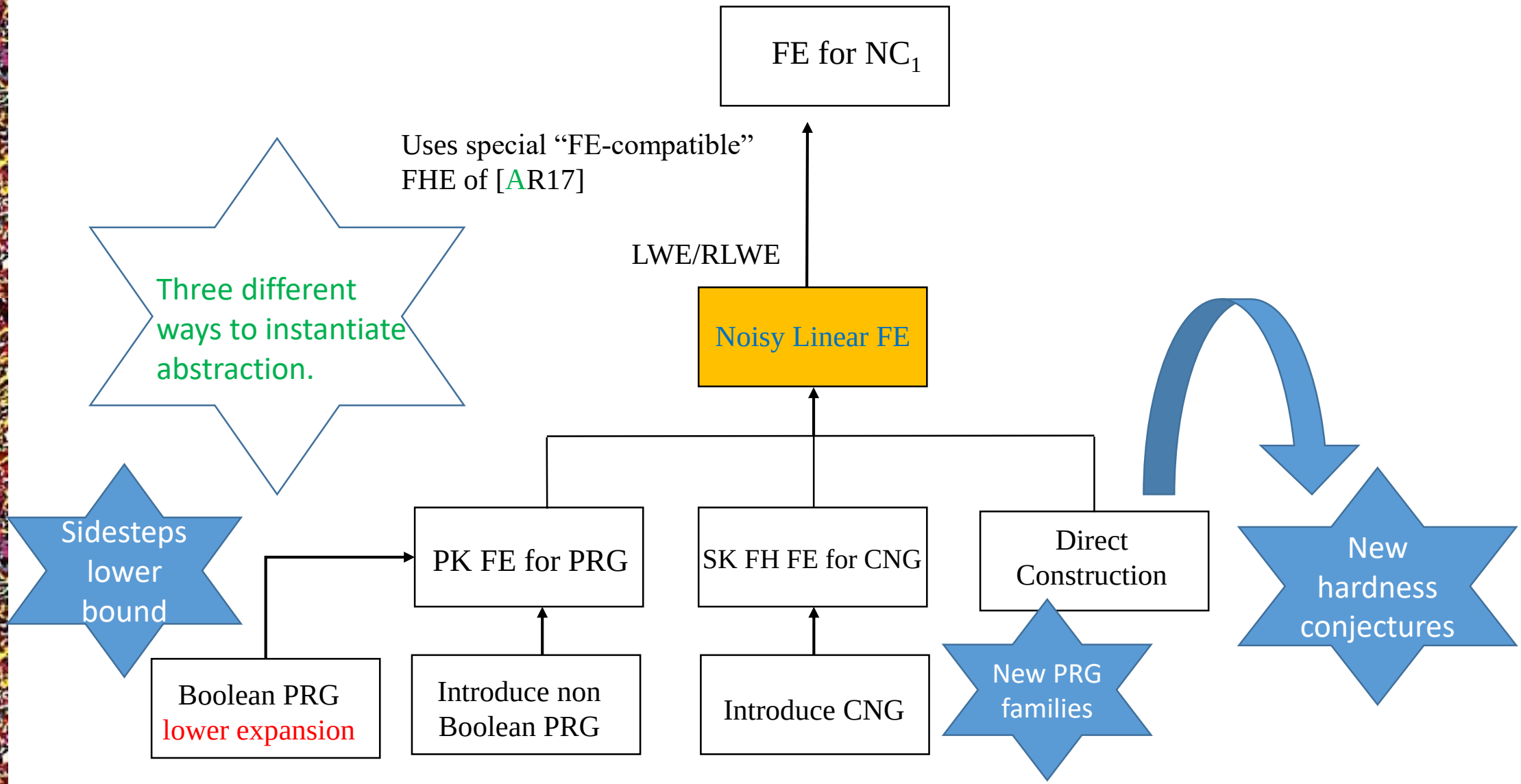
Bootstrapping, determines Exp $\mathcal{F}$



The State of Affairs

- All works use randomizing polynomials for bootstrapping.
- Necessitates Boolean PRG with expansion $\mathcal{E}$.
- Previously, Lin-Tessaro conjecture PRG with degree 2 and expansion $\mathcal{E}$
- LV17, BBKK17 show that degree 2 impossible for expansion $\mathcal{E}$
- Narrow margin of expansion $\mathcal{E}'$ left open by attacks
- But not clear how to use this to build iO.

New Abstraction to build iO [A19]



Fundamental primitive in FE. Addition most basic!

Noisy Linear FE: The right abstraction ?

- Recall Linear FE [ABDP15,ALS16]: Enc(x), Keygen(y), Decrypt to get <x,y>.
- Possible from standard assumptions – DDH, LWE, QR
- Noisy Linear FE : Enc(x), Keygen(y), Decrypt to get <x,y> plus noise
- Where does noise come from?
- What security properties does it need to satisfy?
- Going in circles ?

Noise must satisfy only mild statistical properties

Need not be Boolean, as no need for randomized enc

A key new observation:
Computing a noise term is easier as exact value not important

Weakening requirements on PRGs

Advantage 1: Relax requirements on PRGs

- Boolean: make do with lower expansion $\mathcal{E}'$ which is not ruled out
- Non-Boolean PRGs: Two new classes of randomness generators
  - Need not be Boolean: save degree blowup (caused by arithmetization)
  - Satisfy much weaker property than pseudorandomness
  - May be computable with lower degree
- How to instantiate these PRGs?

Won't have time to talk about this

Advantage 2: Permits Mixed Assumptions

- Noisy linear FE is used black box in bootstrapping
- Bootstrapping uses LWE.
- Noisy linear FE may use any assumption
  - In one instance, mix of pairings and lattices. Uses best of both.

Garuda: Mythical Indian character
Half eagle, half man.

Advantage 3: More Efficient

- Previously FE for degree $\mathcal{L}$ → FE for NC_0 → FE for NC_1 → iO

- Can bootstrap to FE for NC_1 directly.

  Noisy Linear FE → ~~FE for NC_0~~ → FE for NC_1 → iO

  → FE for degree $\mathcal{L}$ → ~~FE for NC_0~~ → FE for NC_1 → iO

- More efficient

Advantage 4: Permits New Direct Constructions

- Previously, all direct constructions use same template:
  - Fix plaintext matrix branching program
  - Add randomization layers: diagonal padding, random scalars, Kilian randomization
  - Encode randomized matrices using one of three mmap families
- All eggs in same basket
- Noisy linear FE can be constructed without multilinear, or even bilinear maps!

Advantage 4: Permits New Direct Constructions

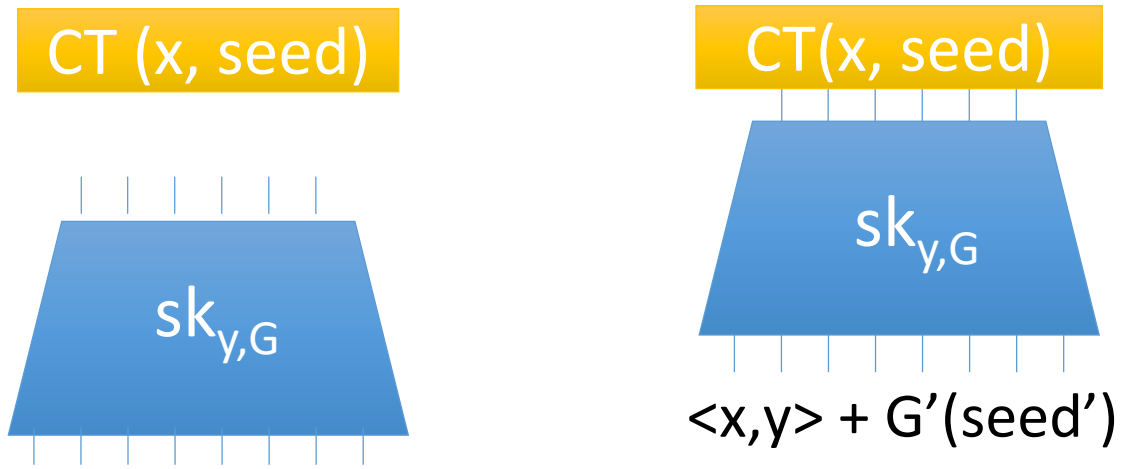
A key new observation: Old grandma advice!

If you cannot have what you want, you must learn to want what you can have

A key new observation: Relax requirement on correctness!

If you cannot compute what you can use, you must learn to use what you can compute

A key new observation: Relax requirement on correctness!



- Only <x,y> needs to be correct! G(seed) is allowed some corruption
- So far: Assume polynomial is PRG and insist on computing it exactly
- Here: Compute whatever can be computed and check if it can satisfy PRG like properties

Advantage 4: Permits New Direct Constructions

- Extend LWE based Linear FE of ALS16 to Noisy Linear FE using new hardness conjectures on lattices.
- Unrelated to multilinear map assumptions (modulo mathematical structure)
- May be more robust. May be post-quantum.
- Much simpler to analyse than mmap based direct constructions: no need for straddling sets, Kilian randomization etc used by all prior work
- First construction of nonlinear FE without any maps.
- Philosophically similar idea used in follow-up [JLMS19]

Bootstrapping Functional Encryption

Noisy Linear FE → ~~FE for NC_0~~ → FE for NC_1 → iO

Ring Learning with Errors Problem

Let ring $R_q = Z_q[x]/<x^n+1>$

DISTRIBUTION 1

Sample s uniformly in R_q

$a_1 , b_1 = a_1 s + err_1$

$a_2 , b_2 = a_2 s + err_2$

$a_m , b_m = a_m s + err_m$

a_i uniform $\in R_q$, $e_i \sim \varphi \in R$

vs

DISTRIBUTION 2

a'_1 , b'_1

a'_2 , b'_2

a'_m , b'_m

a_i, b_i uniform $\in R_q$

Regev Public Key Encryption

Finding short $\vec{e}$ such that $<\vec{a};\vec{e}>=u$ is hard

- SK : $\vec{e}$ PK : $\vec{a},u$
- Encrypt (PK, m) :

$$\vec{c}_0 = \vec{a}\cdot s + \vec{e}rr_1$$

$$c_1 = u\cdot s + err_2 + m\left\lfloor \frac{q}{2} \right\rfloor$$

Pseudorandom By R-LWE

Small only if e is small

- Decrypt (SK) :

$$c_1 - <\vec{e};\vec{c}_0>$$

$$= u\cdot s + err_2 + m\left\lfloor \frac{q}{2} \right\rfloor - u\cdot s - <\vec{e};\vec{e}rr_1>$$

$$= m\left\lfloor \frac{q}{2} \right\rfloor + error$$

Regev Public Key Encryption

- SK : $\vec{e}$ PK : $\vec{a}, u$
- Encrypt (PK, m) :

$$\vec{c}_0 = \vec{a} \cdot s + \vec{e}rr_1$$

$$c_1 = u \cdot s + err_2 + m\left\lfloor \frac{q}{2} \right\rfloor$$

Randomness encoding

Message encoding

- Decrypt (SK) :

$$c_1 - < \vec{e}; \vec{c}_0 >$$

$$= u \cdot s + err_2 + m\left\lfloor \frac{q}{2} \right\rfloor - u \cdot s - < \vec{e}; \vec{e}rr_1 >$$

$$= m\left\lfloor \frac{q}{2} \right\rfloor + error$$

What's special about this PKE?

Lends Itself to Fully Homomorphic Encryption!

Symmetric key FHE for Quadratic Polynomials (BV11)

s: secret key

Encrypt (s, x_1, x_2):

Sample u_1, u_2 randomly in ring. Sample err_1, err_2.

Compute :

$$c_1 = u_1 s + err_1 + x_1$$

$$c_2 = u_2 s + err_2 + x_2$$

Evaluate (c_1, c_2, $f = x_1 x_2$):

Want: Use c_1, c_2 to compute product ciphertext c_{12} that encrypts $x_1 x_2$

FHE Evaluation

We may write:

$$x_1 \approx c_1 - u_1 s$$

$$x_2 \approx c_2 - u_2 s$$

$$\therefore x_1 x_2 \approx c_1 c_2 - (c_1 u_2 + c_2 u_1) s + u_1 u_2 s^2$$

Let $c^{mult} = (c_1\, c_2, \quad c_1\, u_2 + c_2\, u_1, \quad u_1 u_2)$

Given secret key s, and ciphertext c^{mult}, decryptor can recover message up to noise.

Onwards to Functional Encryption

- FHE secret key s permits decryptor to also decrypt original messages x_1 and x_2.
- Wish to constrain decryption so that key holder learns x_1x_2 but not individual x_1 and x_2

Recall….

Regev PKE Ciphertext :

$$\vec{c} = \vec{a} \cdot s + \vec{e}rr_1$$

$$c_1 = u_1 \cdot s + err_2 + x_1$$

$$c_2 = u_2 \cdot s + err_2 + x_2$$

Observe:

- Ciphertext can be split into randomness carrier “c” and message carrier c_1, c_2.

- Message carrier exactly resembles FHE Symmetric key CT

The Hope

- Switch View to FHE:
  - Interpret PKE message carrier components as FHE symmetric key ciphertexts
- FHE Computation:
  - Evaluate f on encrypted data using FHE.
  - Recover encryption of f(x) $c_f = u_f \cdot s + err_f + f(\vec{x})$
- Switch view back to Regev PKE:

  Given randomness carrier and message carrier for f(x), decrypt as in original Regev PKE

$$\vec{c} = \vec{a} \cdot s + \vec{e}rr_1$$

$$c_f = u_f \cdot s + err_f + f(\vec{x})$$

The News (Good and Bad)

- The Bad: Too good to be true
- Want FHE computation to result in a CT

$$c_f = u_f \cdot s + err_f + f(\vec{x})$$

KeyGen needs but cannot know LWE label

- Instead, CT looks like

$$c_f = u_{f,ct} \cdot s + err_f + f(\vec{x})$$

- The Good: This blueprint works for linear functions
- Given CT(x), SK(y), decryption outputs <x,y>
- We'll leverage linear functions to support deeper circuits

Generalizing to Quadratic (AR17)

- Recall FHE multiplication:

$$x_1 \approx c_1 - u_1 s$$

$$x_2 \approx c_2 - u_2 s$$

$$\therefore x_1 x_2 \approx c_1 c_2 - (c_1 u_2 + c_2 u_1) s + u_1 u_2 s^2$$

- What if we group the terms differently?

$$\therefore x_1 x_2 \approx c_1 c_2 - u_2 (c_1 s) - u_1 (c_2 s) + u_1 u_2 (s^2)$$

Functional Encryption for Quadratic polynomials P(x) = x_1x_2

Enc(mpk, $\vec{x} = (x_1 \ldots . x_n)$):

$\{c_i = u_i \cdot s + err_i + x_i\}_{i\in[n]}$

LinearFE.Enc$(s^2, c_1\ s, \ldots . c_n\ s)$

Kgen(msk, P):

LinearFE.KGen $(u_1u_2, -u_2, -u_1, 0, .., 0)$

Dec(sk_P, ct_x):

LinearFE.Dec

$$< (s^2, c_1\ s, \ldots . c_n\ s), (u_1u_2, -u_2, -u_1, 0, .., 0) >$$

$$= u_1u_2(s^2) - u_2(c_1s) - u_1(c_2s)$$

$$= c_1c_2 - u_2c_1s - u_1c_2s + u_1u_2s^2 - c_1c_2$$

$$\approx x_1x_2 - c_1c_2$$

Decryptor can compute c_1c_2 itself.

Is this secure?

- Attack: Easily recover s given exact linear equation

$$u_2(c_1 s) + u_1(c_2 s) + u_1 u_2(s^2)$$

- Intuition for fix: Exact linear equations trivial, noisy linear equations intractable

Motto: Add noise!

- Replace linear FE by noisy linear FE.
- This is not a proof!
- New proof technique to show this works

Wrapping it up

- Generalizes to NC_1 via induction
  - Very technical!
  - Need encryptor to provide some extra encodings as “advice”
  - Need to only compute linear function plus noise, i.e. noisy linear FE
- Can use FE for PRG or direct construction to generate noise
- Concurrent work by AJS18 identify similar classes of PRG, incomparable results
- Follow up work by LM18 improves assumption on PRG by handling leakage caused by polynomial bounded PRG

Thank You for your attention ☺

Image Credits: Jackson Pollock, who solves similar problems in a different space!