# On the Fine-Grained Hardness of Lattice Problems

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Stephens-Davidowitz

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Bennett

Alexander
Golovnev

Divesh Aggarwal

# Game Plan

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- Motivation

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  - How secure is lattice-based crypto?

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- Motivation
  - How secure is lattice-based crypto?
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- Summary of results
- Fine-grained hardness of CVP
- Fine-grained hardness of SVP
- Where do we go from here?

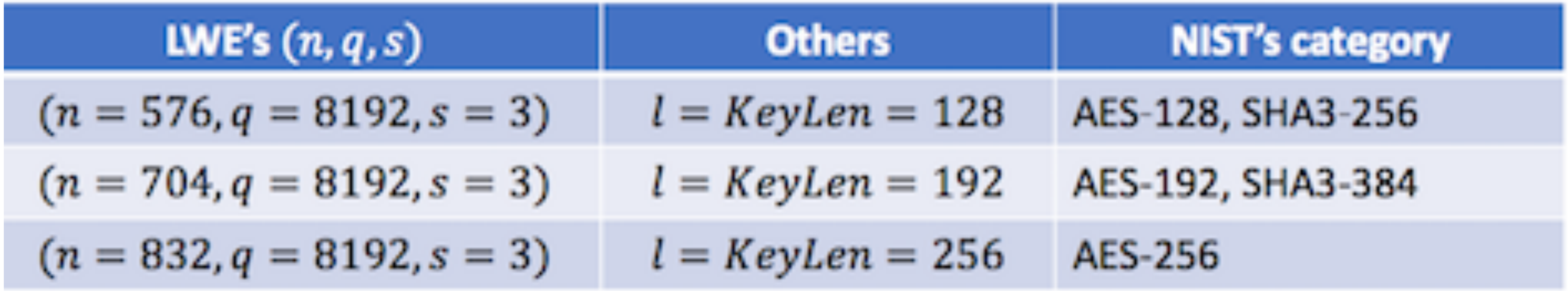
# Act I:

# How confident are we in our security claims?

THE DEVIL'S ADVOCATE

Adversary's
ADVOCATE

# Quantitative Security Claims



| LWE's $(n, q, s)$ | Others | NIST's category |
|---|---|---|
| $(n = 576, q = 8192, s = 3)$ | $l = KeyLen = 128$ | AES-128, SHA3-256 |
| $(n = 704, q = 8192, s = 3)$ | $l = KeyLen = 192$ | AES-192, SHA3-384 |
| $(n = 832, q = 8192, s = 3)$ | $l = KeyLen = 256$ | AES-256 |

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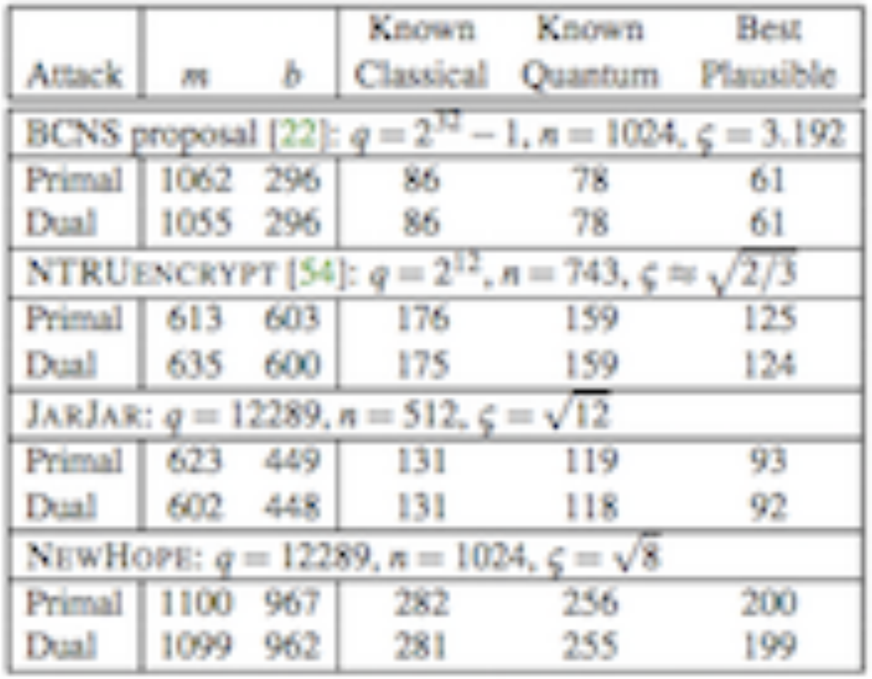
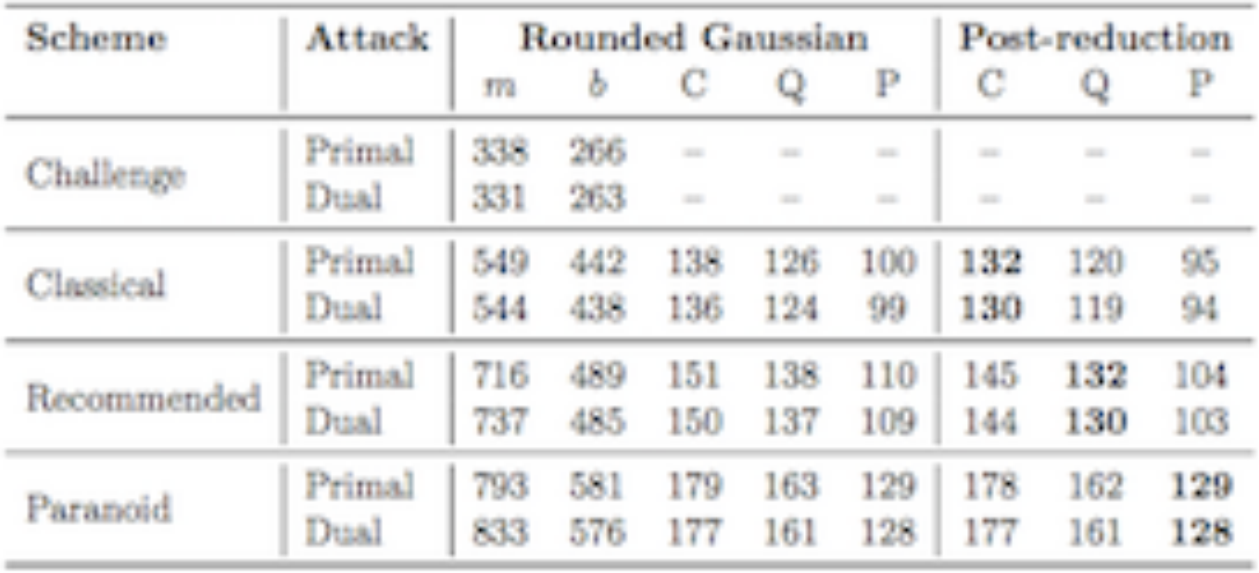
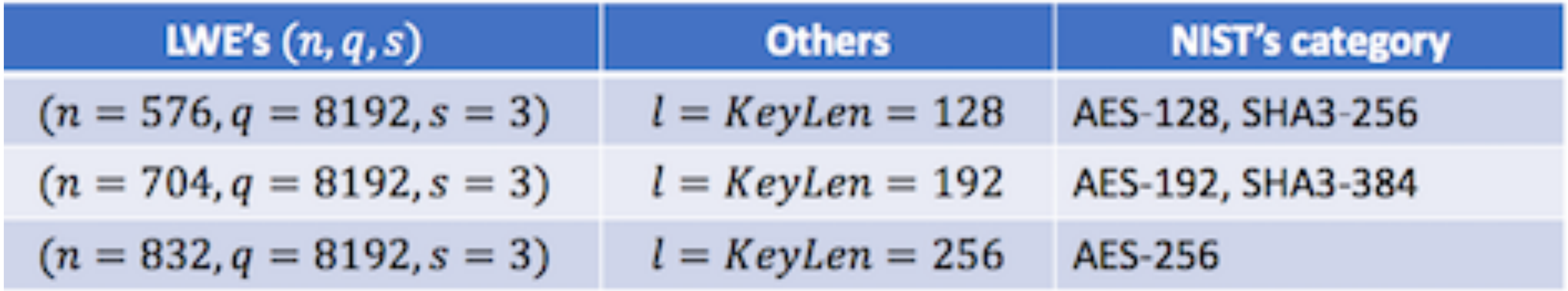


| Attack | $m$ | $b$ | Known Classical | Known Quantum | Best Plausible |
|---|---|---|---|---|---|
| BCNS proposal [22]: $q = 2^{32} - 1$, $n = 1024$, $\varsigma = 3.192$ | | | | | |
| Primal | 1062 | 296 | 86 | 78 | 61 |
| Dual | 1055 | 296 | 86 | 78 | 61 |
| NTRUENCRYPT [54]: $q = 2^{11}$, $n = 743$, $\varsigma \approx \sqrt{2/3}$ | | | | | |
| Primal | 613 | 603 | 176 | 159 | 125 |
| Dual | 635 | 600 | 175 | 159 | 124 |
| JARJAR: $q = 12289$, $n = 512$, $\varsigma = \sqrt{12}$ | | | | | |
| Primal | 623 | 449 | 131 | 119 | 93 |
| Dual | 602 | 448 | 131 | 118 | 92 |
| NEWHOPE: $q = 12289$, $n = 1024$, $\varsigma = \sqrt{8}$ | | | | | |
| Primal | 1100 | 967 | 282 | 256 | 200 |
| Dual | 1099 | 962 | 281 | 255 | 199 |



| Scheme | Attack | Rounded Gaussian | | | | | Post-reduction | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $m$ | $b$ | C | Q | P | C | Q | P |
| Challenge | Primal | 338 | 266 | – | – | – | – | – | – |
| | Dual | 331 | 263 | – | – | – | – | – | – |
| Classical | Primal | 549 | 442 | 138 | 126 | 100 | **132** | 120 | 95 |
| | Dual | 544 | 438 | 136 | 124 | 99 | **130** | 119 | 94 |
| Recommended | Primal | 716 | 489 | 151 | 138 | 110 | 145 | **132** | 104 |
| | Dual | 737 | 485 | 150 | 137 | 109 | 144 | **130** | 103 |
| Paranoid | Primal | 793 | 581 | 179 | 163 | 129 | 178 | 162 | **129** |
| | Dual | 833 | 576 | 177 | 161 | 128 | 177 | 161 | **128** |



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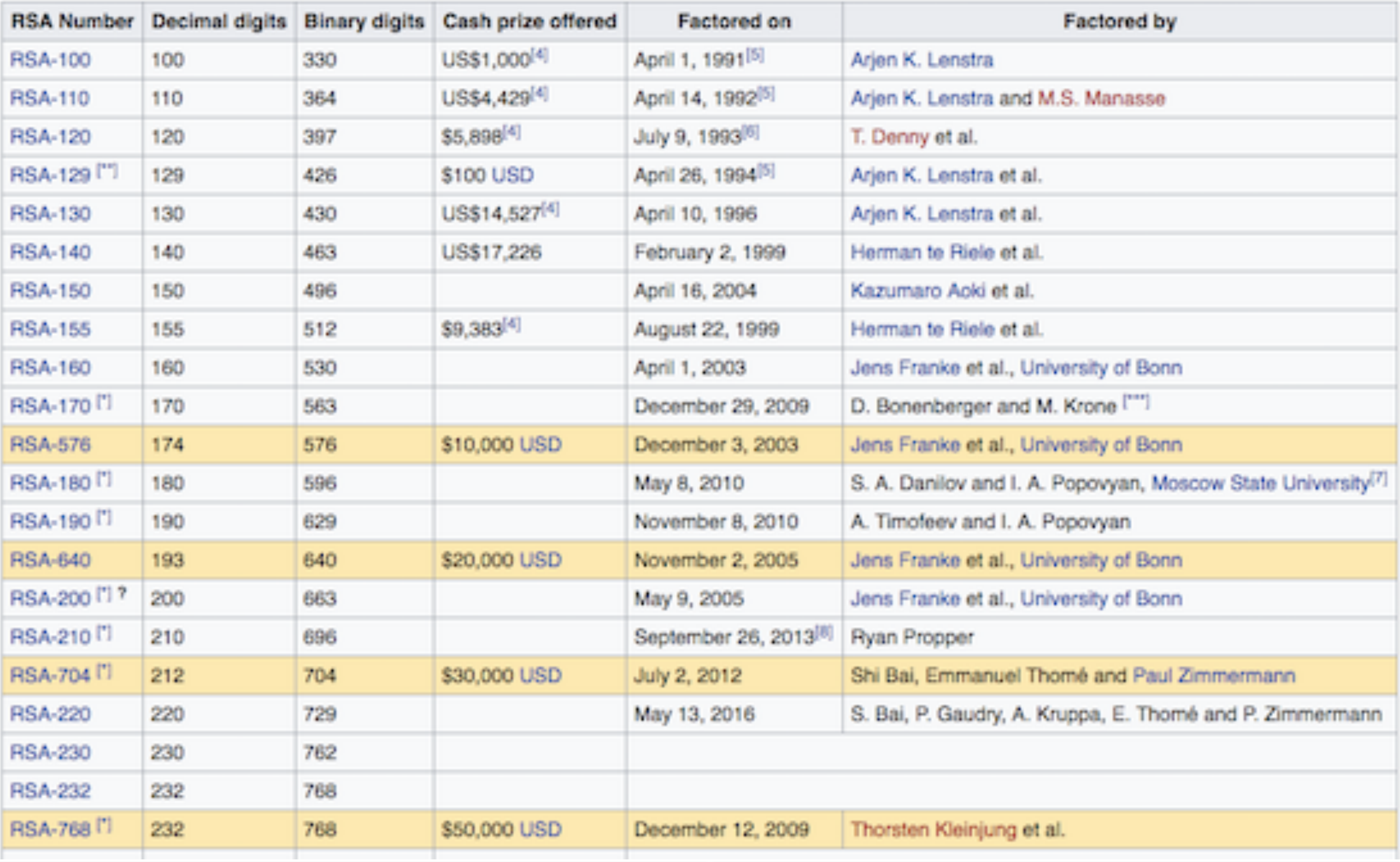

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| RSA-120 | 120 | 397 | $5,898[4] | July 9, 1993[6] | T. Denny et al. |
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| RSA-130 | 130 | 430 | US$14,527[4] | April 10, 1996 | Arjen K. Lenstra et al. |
| RSA-140 | 140 | 463 | US$17,226 | February 2, 1999 | Herman te Riele et al. |
| RSA-150 | 150 | 496 | | April 16, 2004 | Kazumaro Aoki et al. |
| RSA-155 | 155 | 512 | $9,383[4] | August 22, 1999 | Herman te Riele et al. |
| RSA-160 | 160 | 530 | | April 1, 2003 | Jens Franke et al., University of Bonn |
| RSA-170 [*] | 170 | 563 | | December 29, 2009 | D. Bonenberger and M. Krone [***] |
| RSA-576 | 174 | 576 | $10,000 USD | December 3, 2003 | Jens Franke et al., University of Bonn |
| RSA-180 [*] | 180 | 596 | | May 8, 2010 | S. A. Danilov and I. A. Popovyan, Moscow State University[7] |
| RSA-190 [*] | 190 | 629 | | November 8, 2010 | A. Timofeev and I. A. Popovyan |
| RSA-640 | 193 | 640 | $20,000 USD | November 2, 2005 | Jens Franke et al., University of Bonn |
| RSA-200 [*] ? | 200 | 663 | | May 9, 2005 | Jens Franke et al., University of Bonn |
| RSA-210 [*] | 210 | 696 | | September 26, 2013[8] | Ryan Propper |
| RSA-704 [*] | 212 | 704 | $30,000 USD | July 2, 2012 | Shi Bai, Emmanuel Thomé and Paul Zimmermann |
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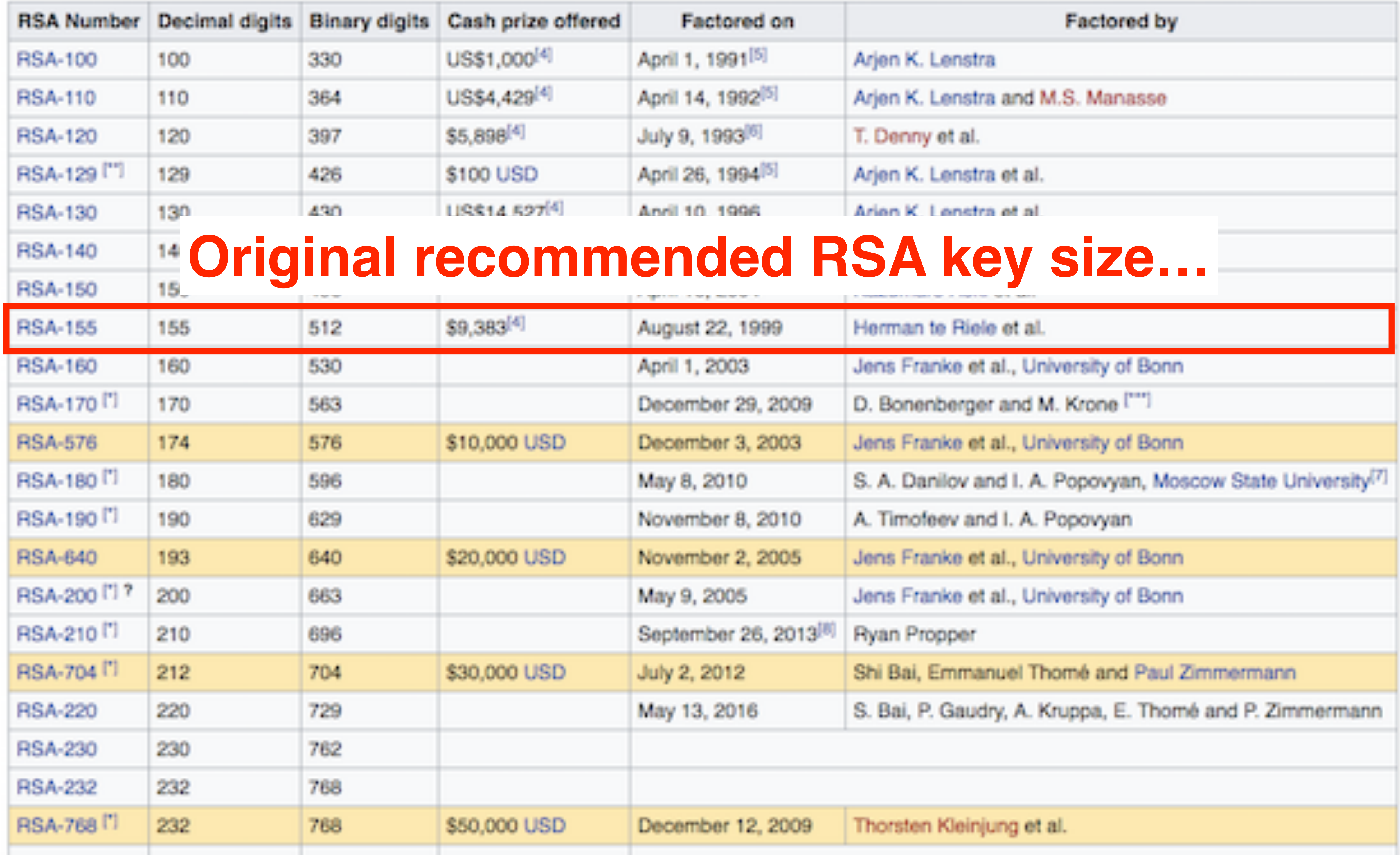

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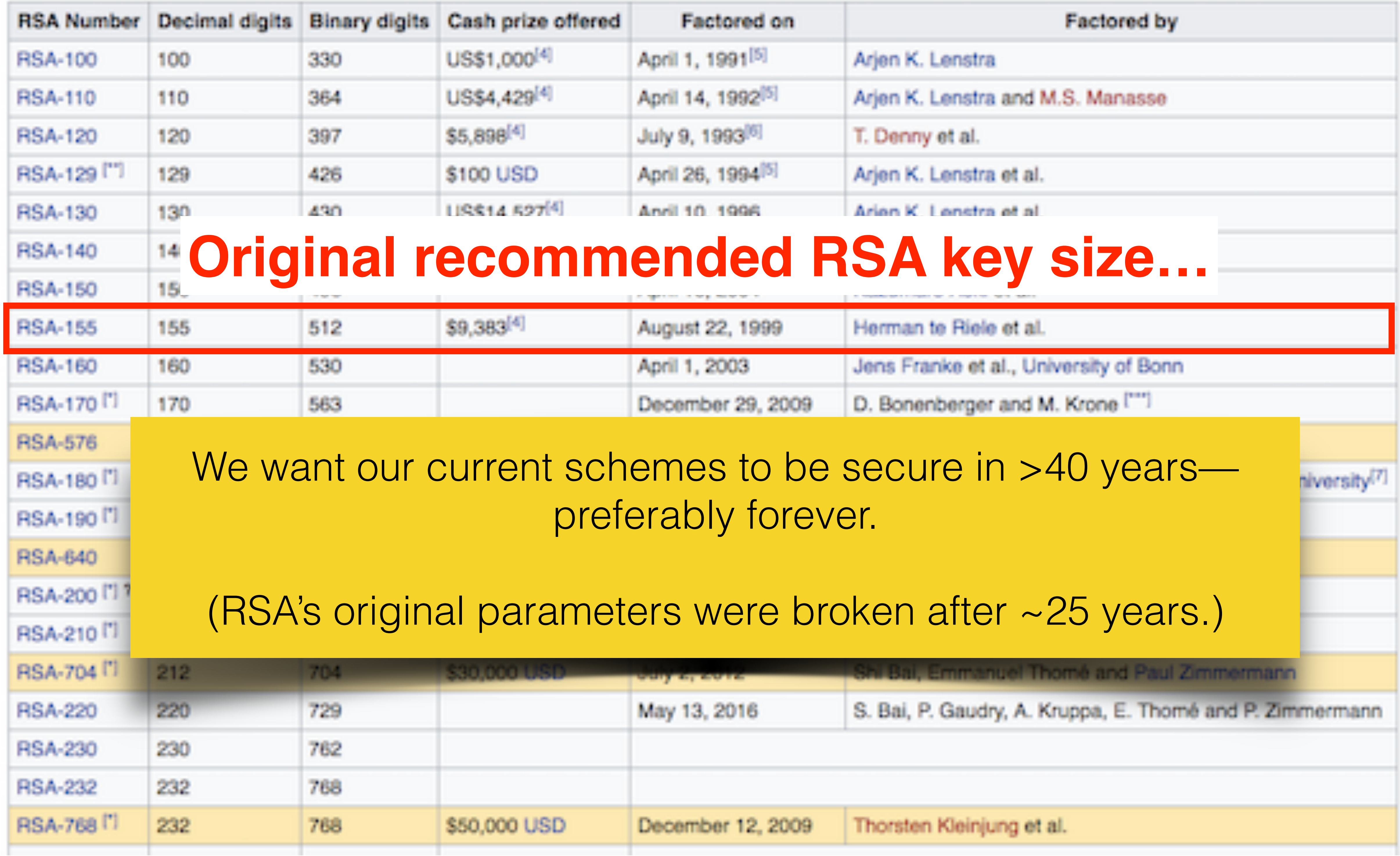

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**Original recommended RSA key size…**

We want our current schemes to be secure in >40 years—preferably forever.

(RSA's original parameters were broken after ~25 years.)

# Security of Lattice-Based Crypto (a caricature)

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Crypto
(LWE)

$\mathbf{A} \sim \mathbb{Z}_q^{n \times m}$
Is there $\mathbf{s} \in \mathbb{Z}^n$
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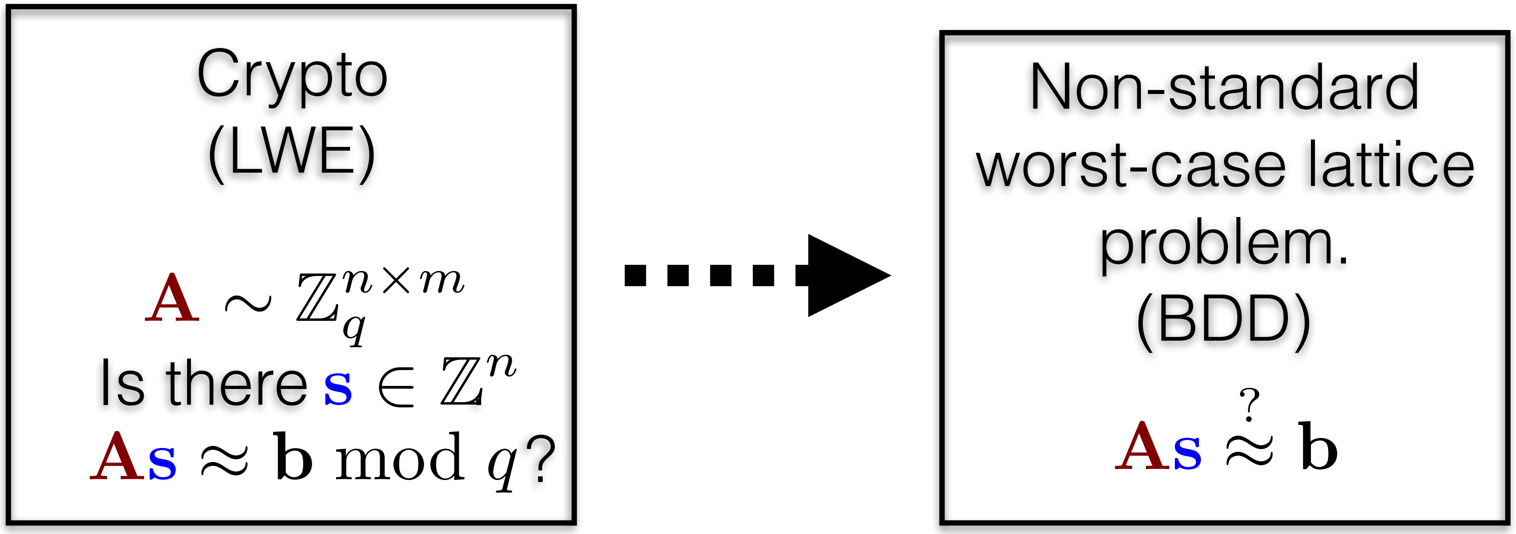

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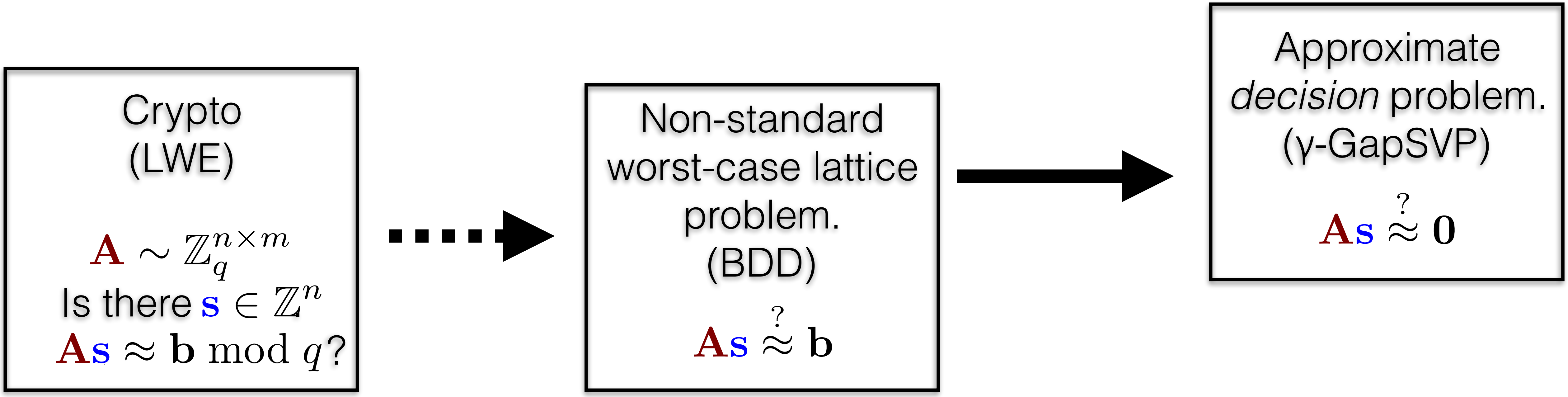

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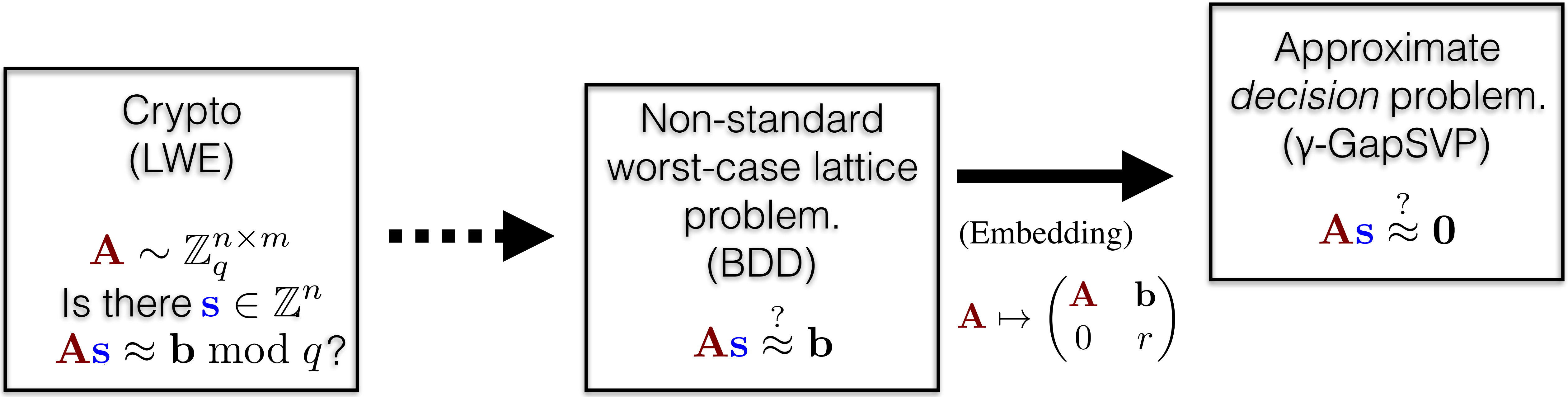

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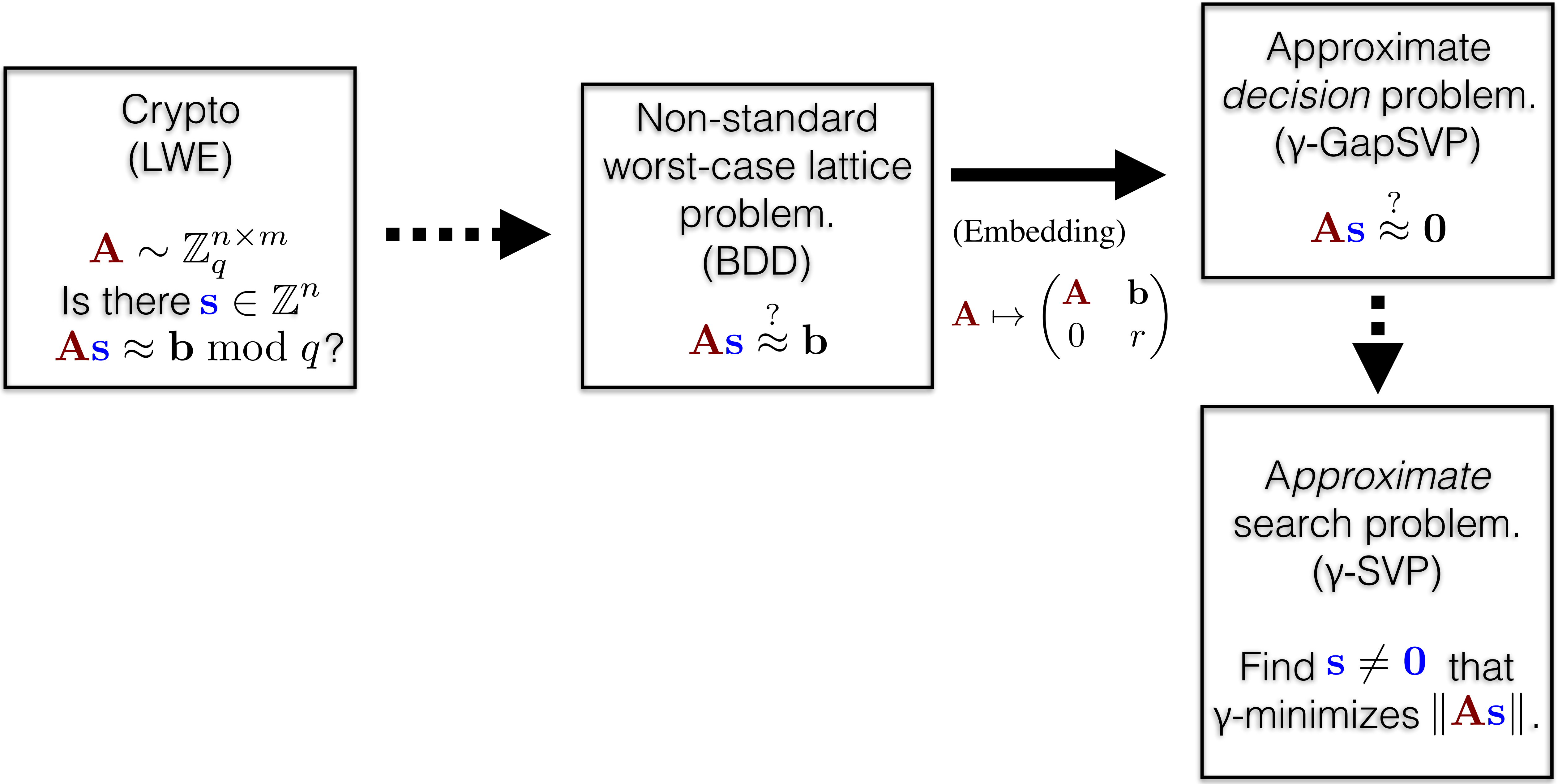

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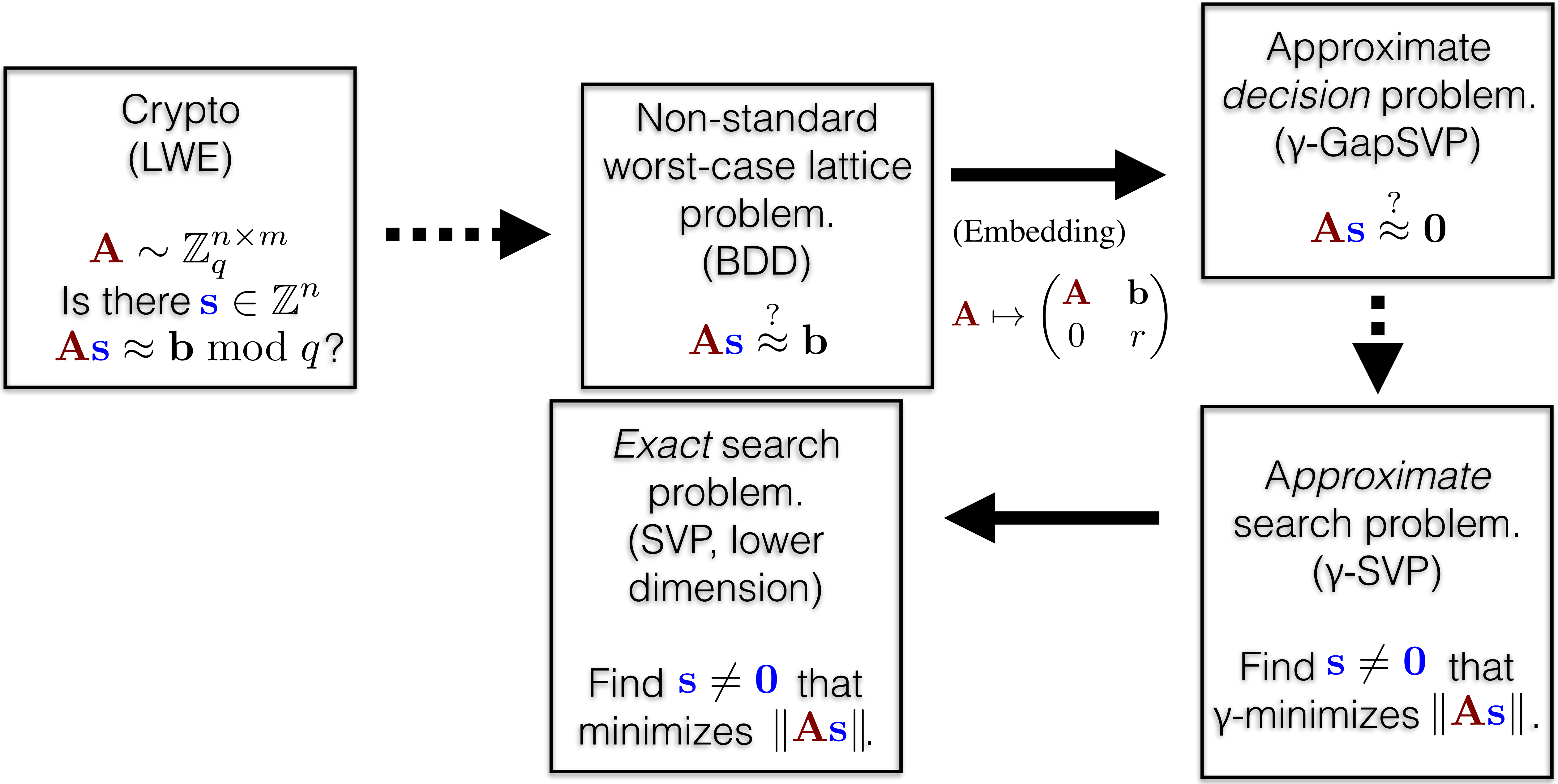

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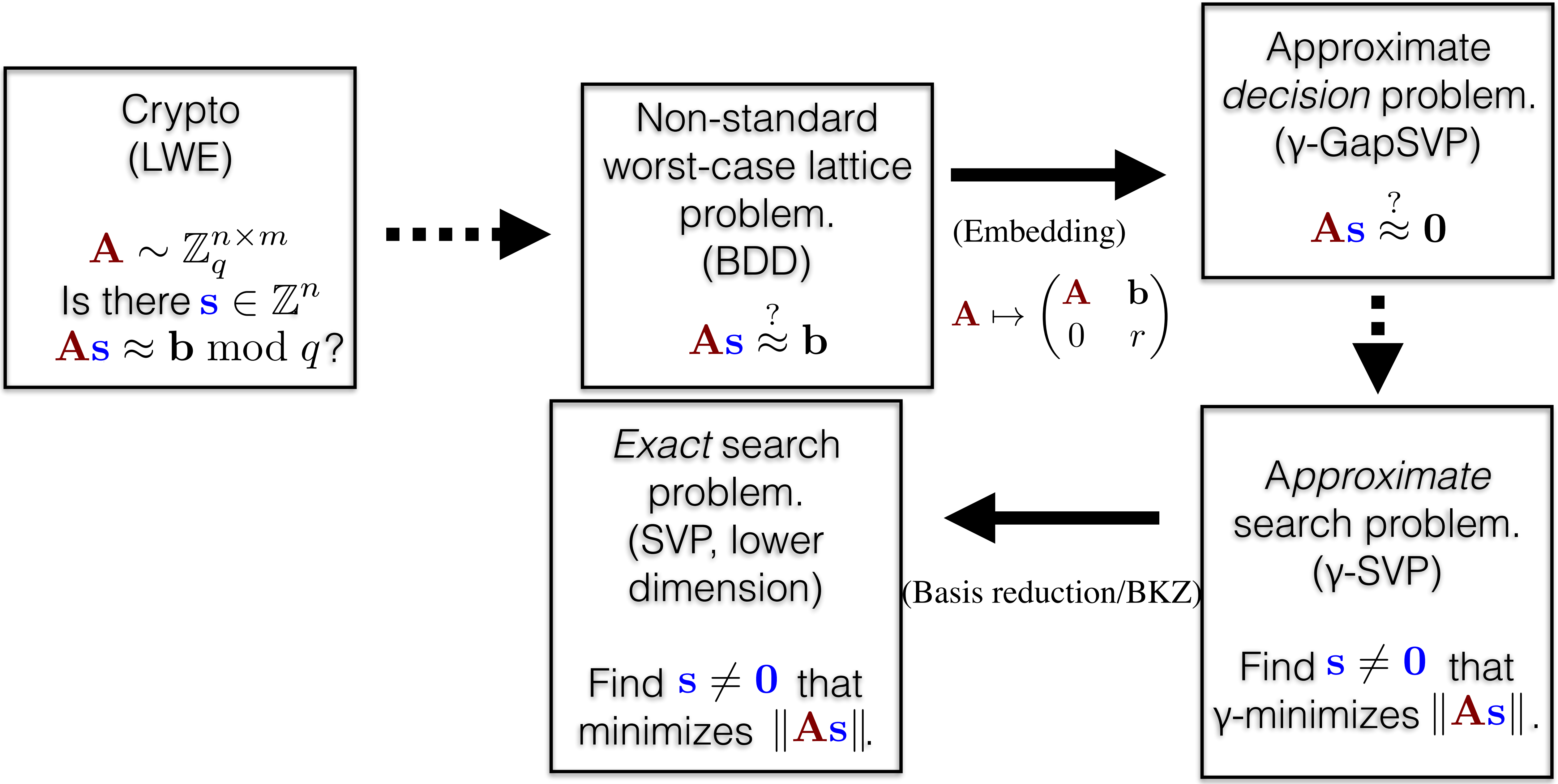

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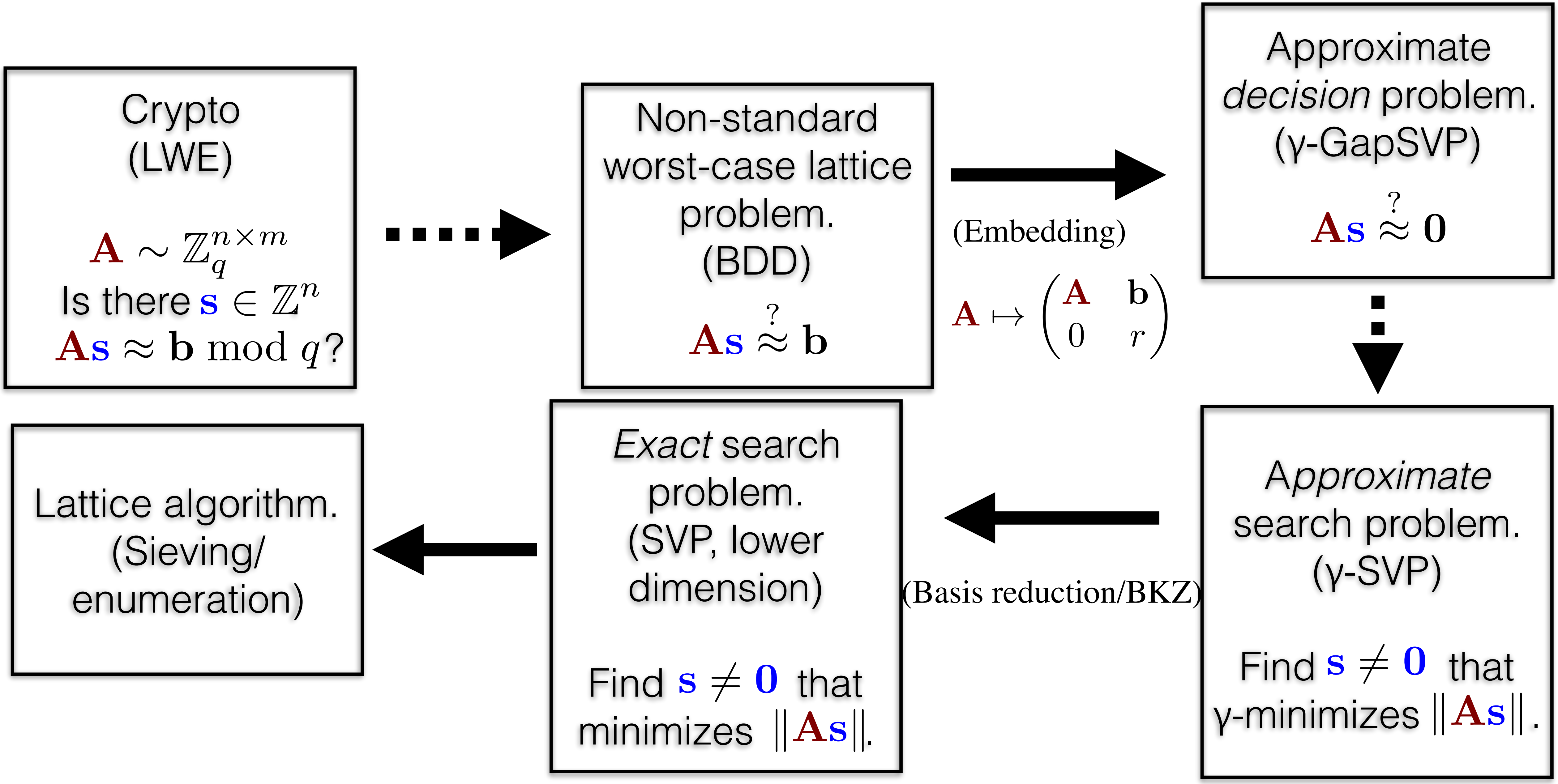

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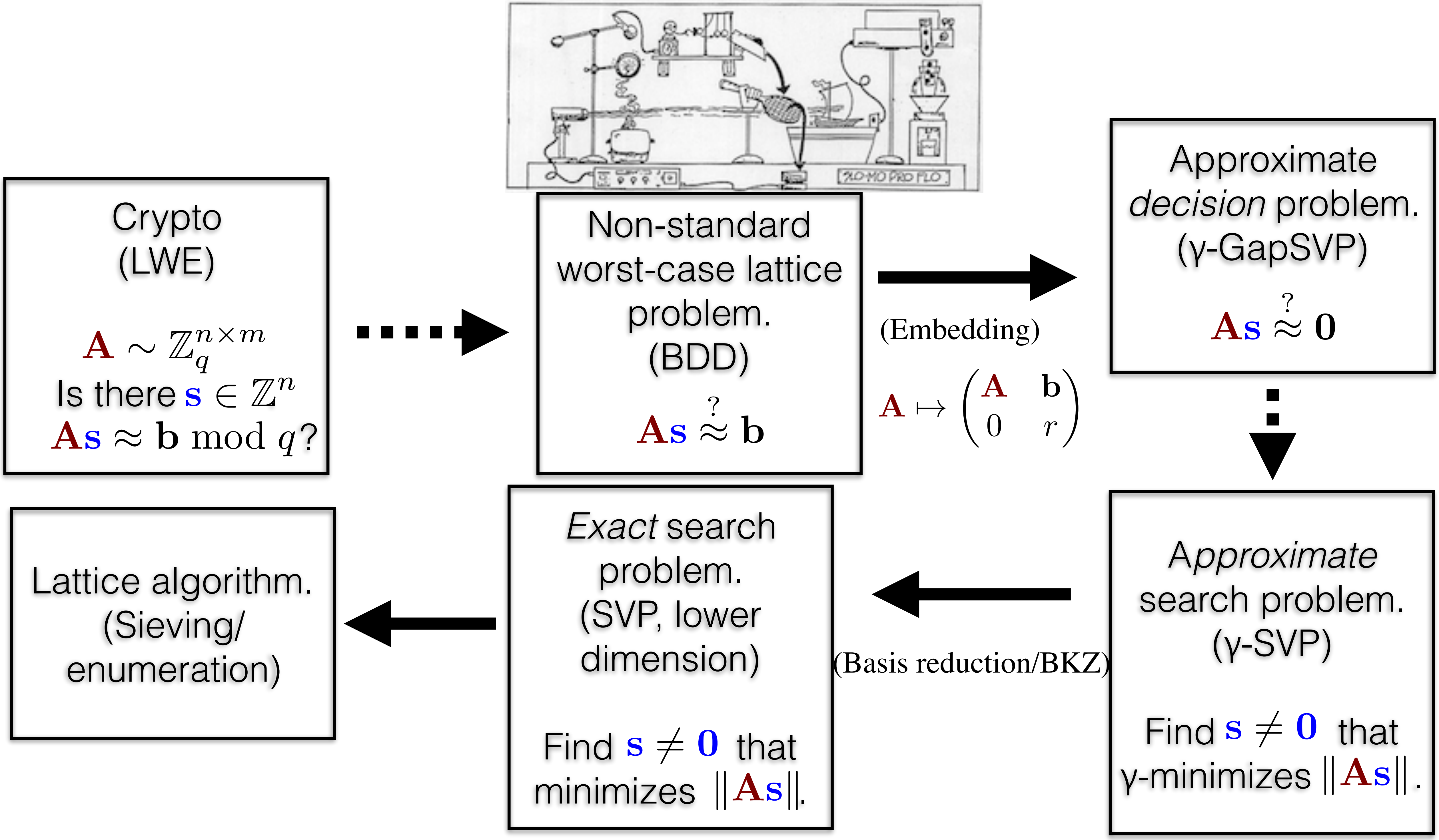

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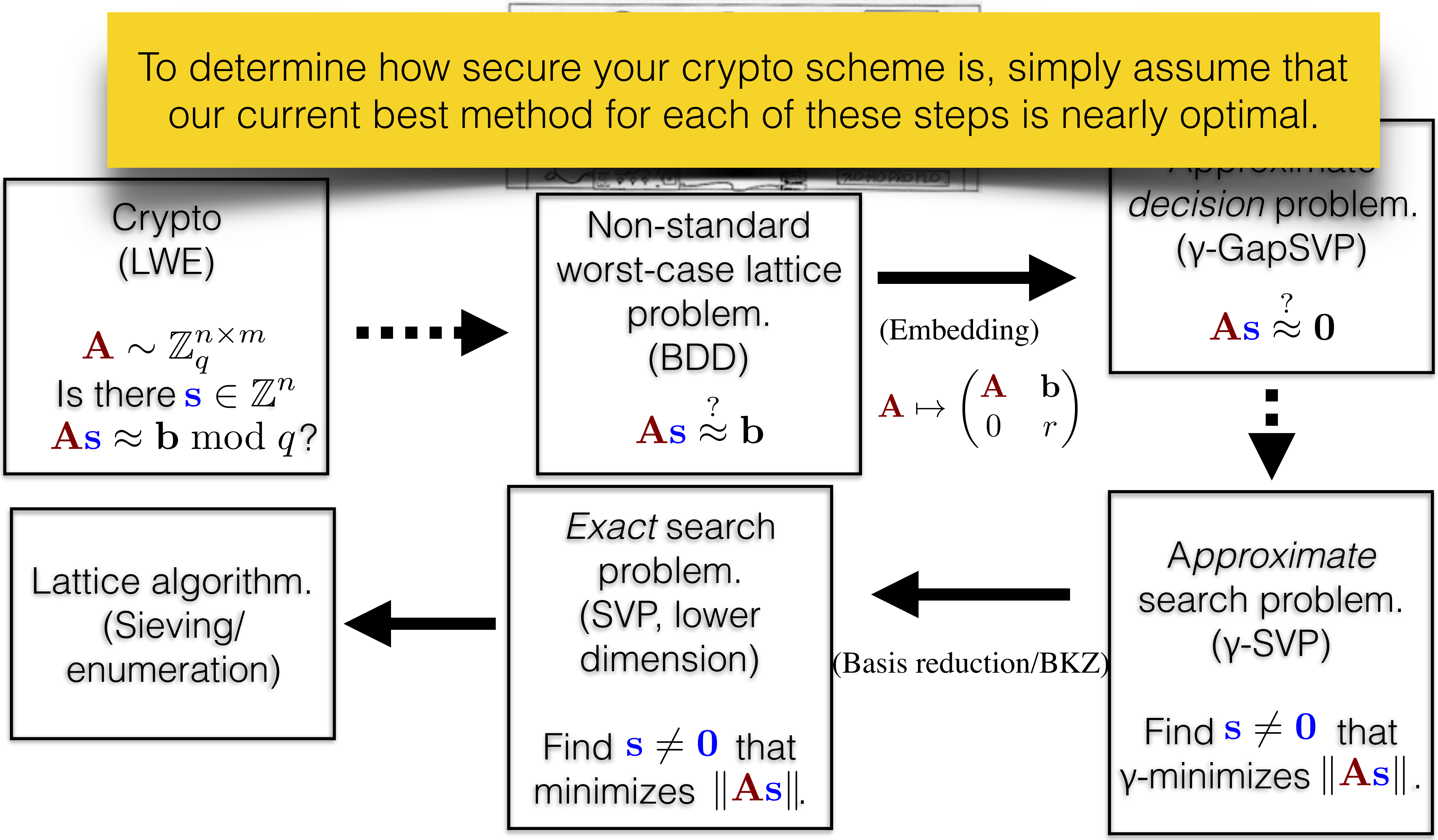

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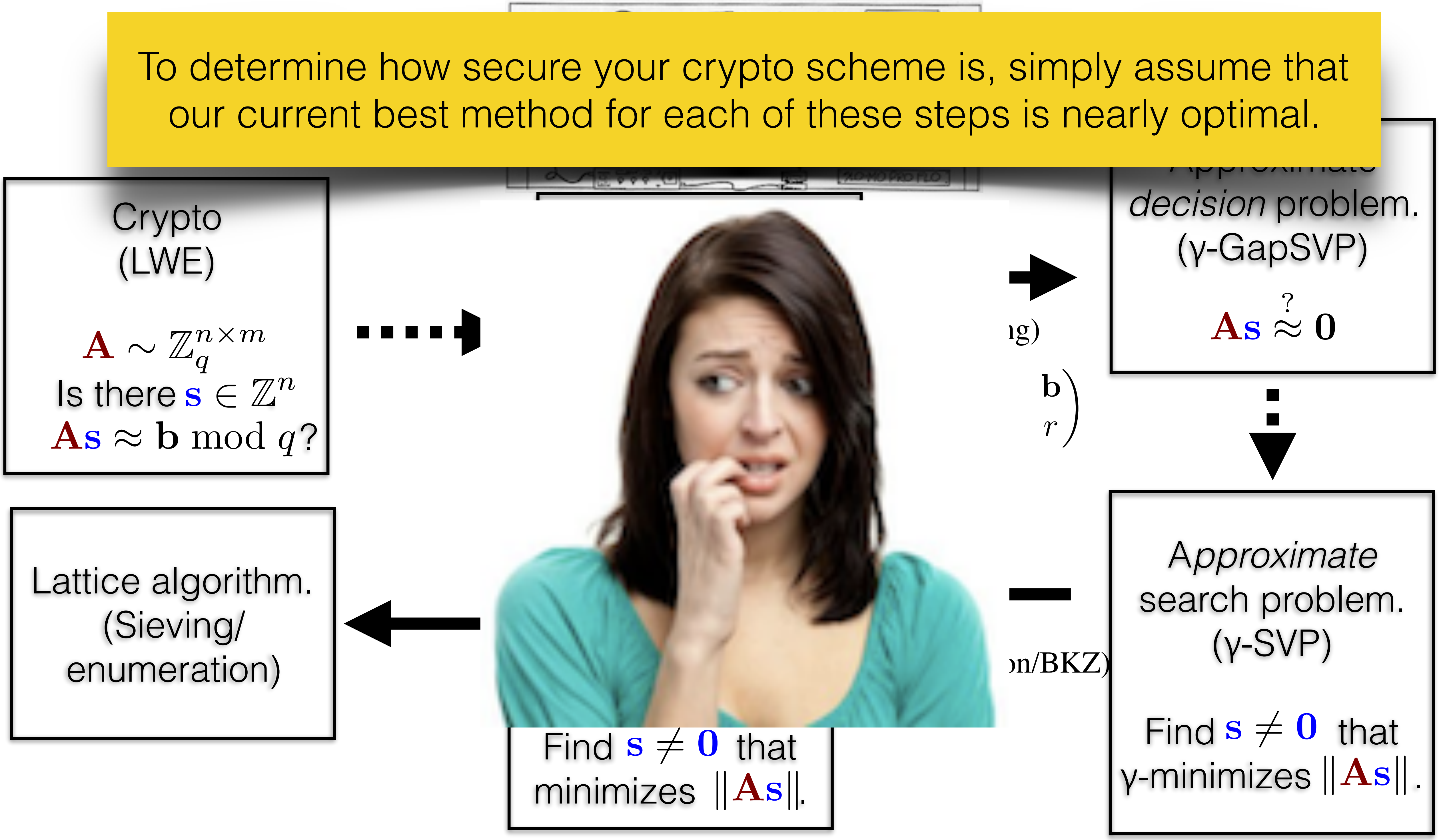

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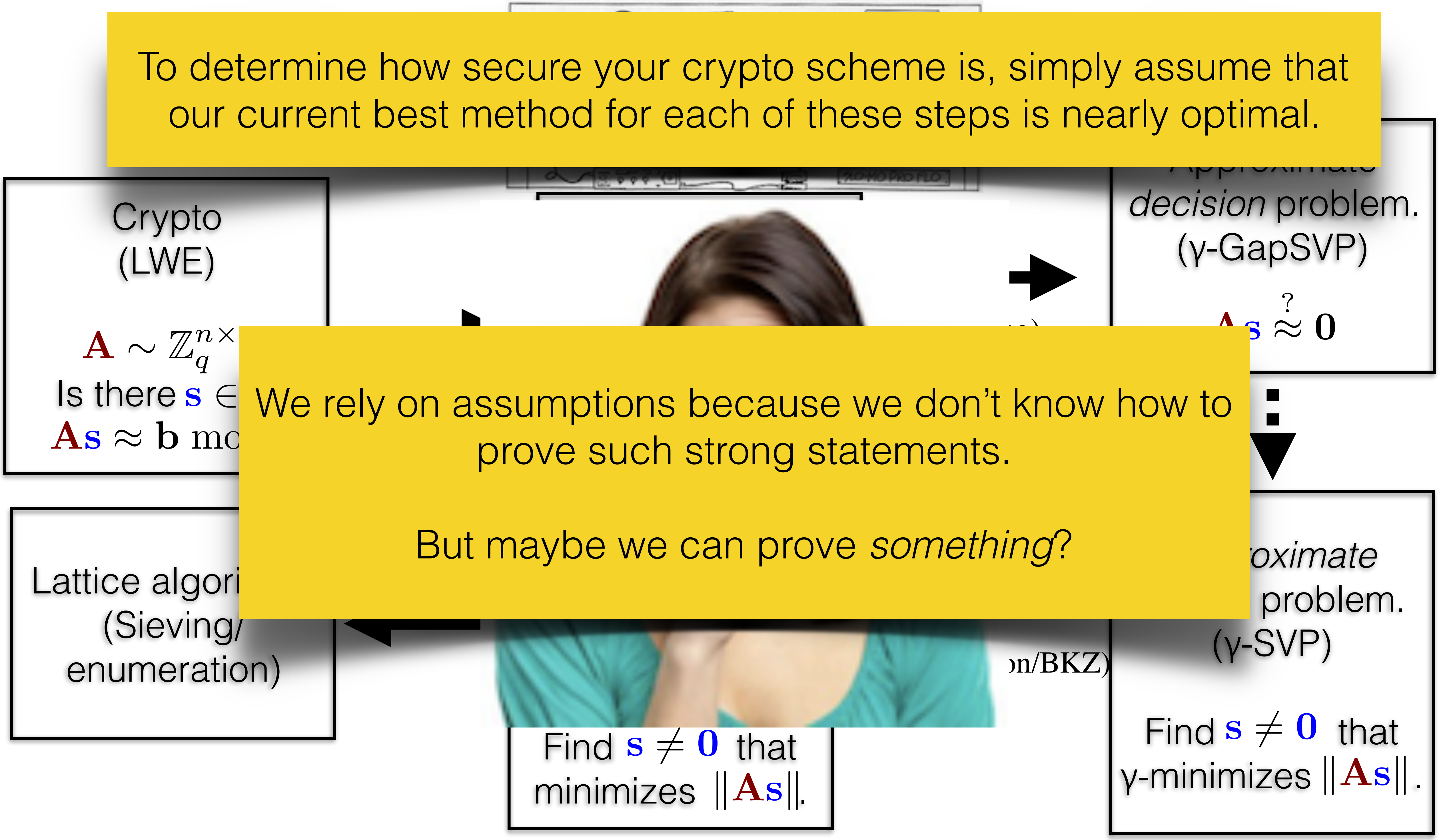

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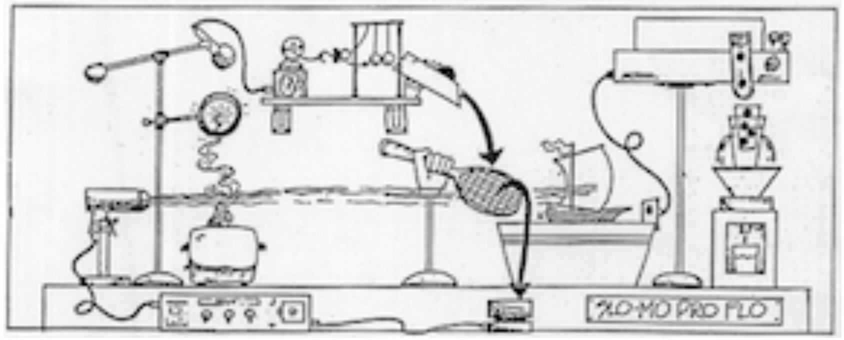
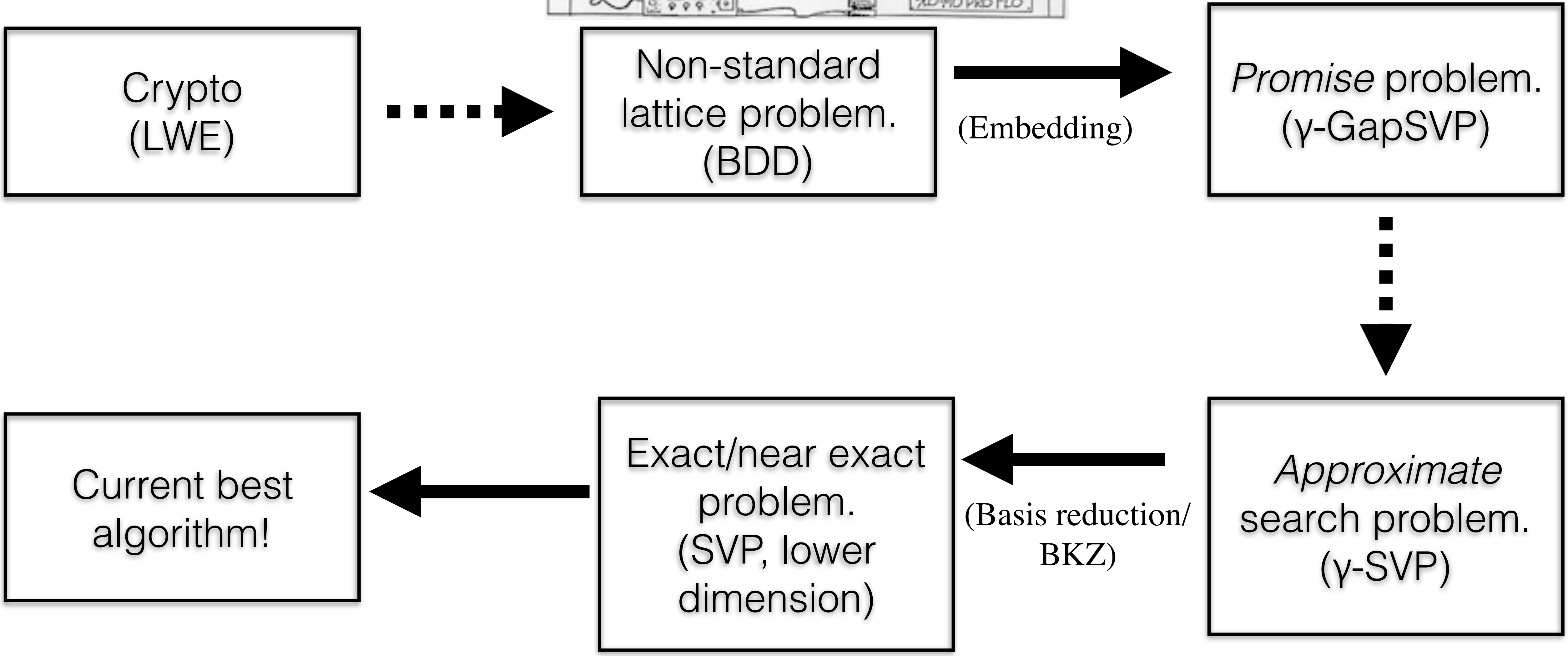
Lattice algori (Sieving/ enumeration)

We rely on assumptions because we don’t know how to prove such strong statements.

But maybe we can prove *something*?

# Security of Lattice-Based Crypto





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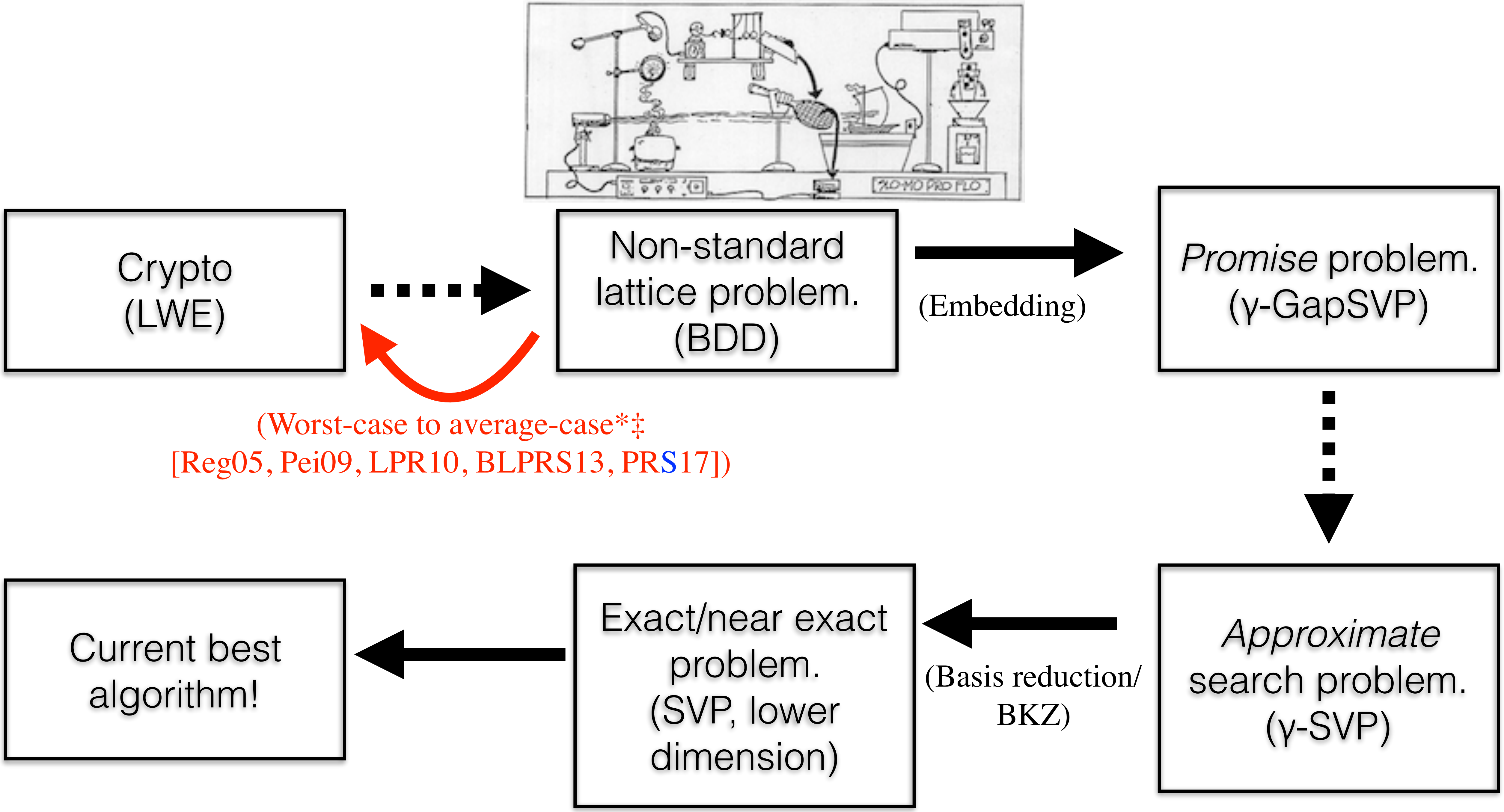

Crypto (LWE)

Non-standard lattice problem. (BDD)

(Worst-case to average-case*‡ [Reg05, Pei09, LPR10, BLPRS13, PRS17])

(Embedding)

*Promise* problem. (γ-GapSVP)

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Current best algorithm!

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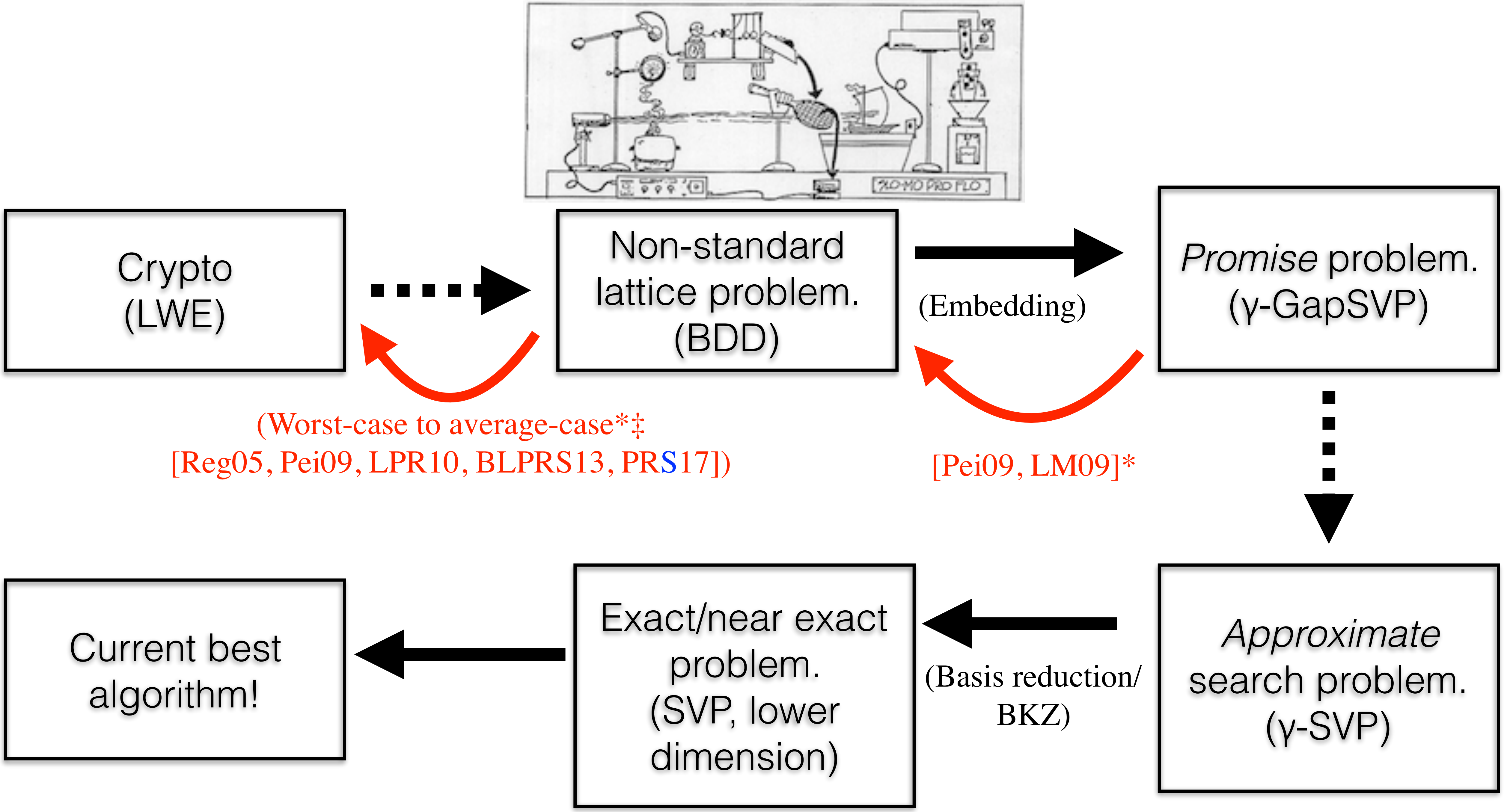

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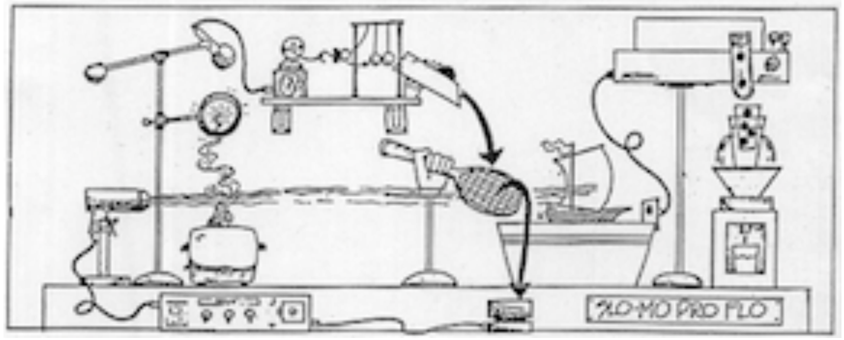

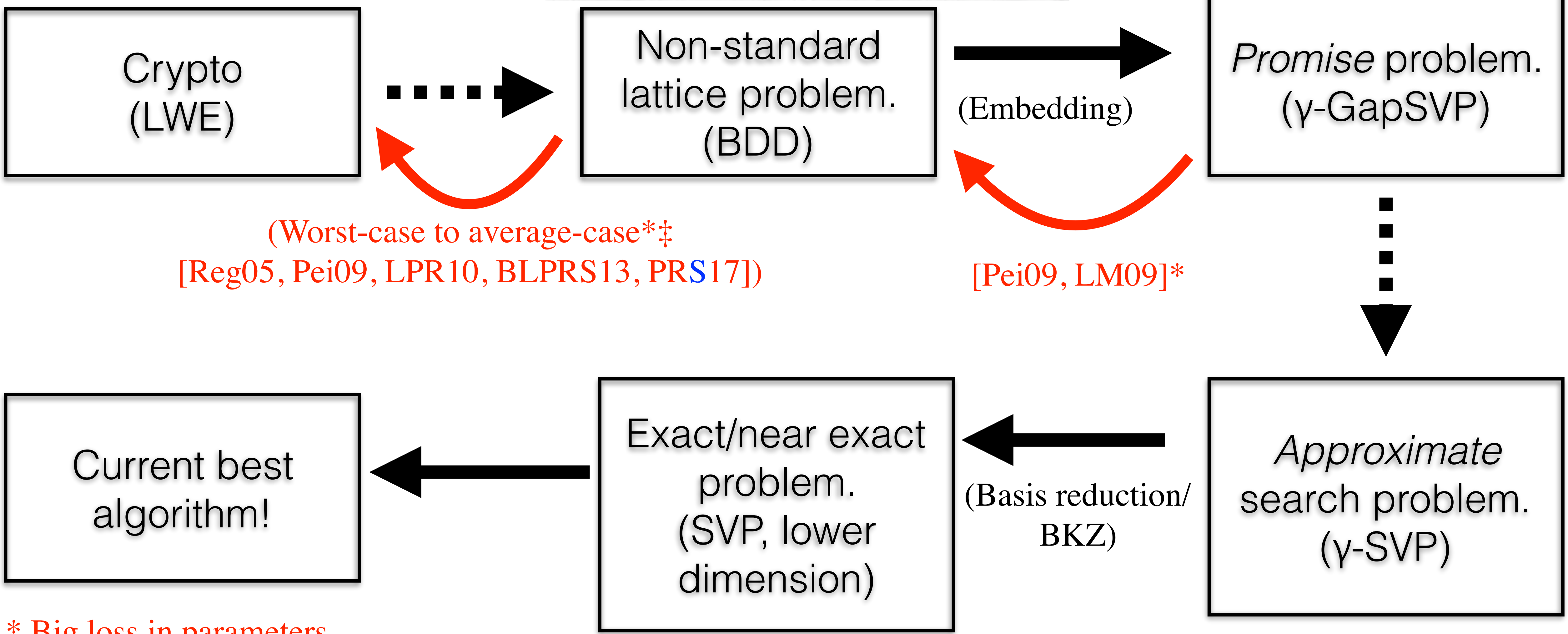

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‡ Doesn't apply for many practical schemes.

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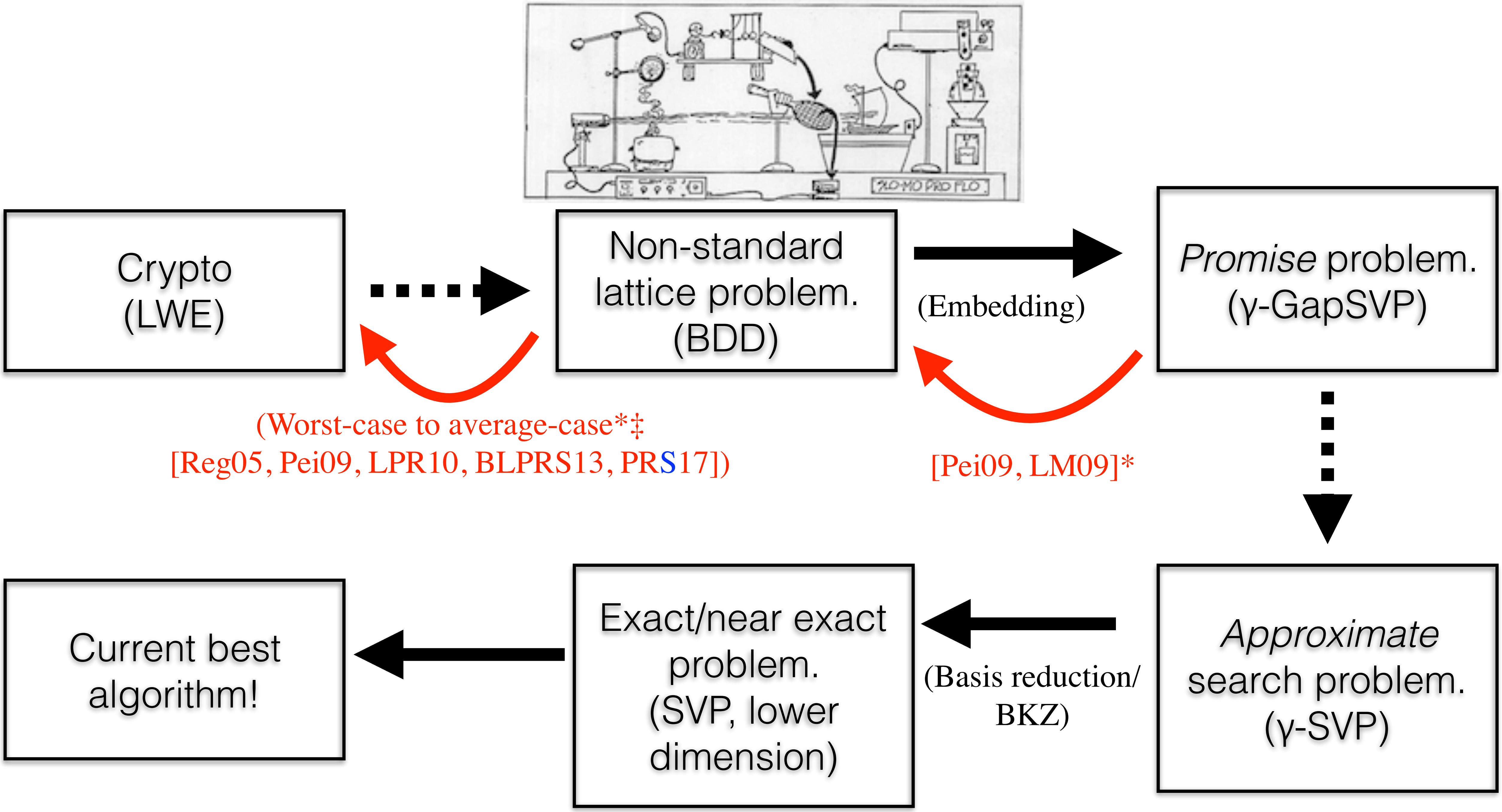

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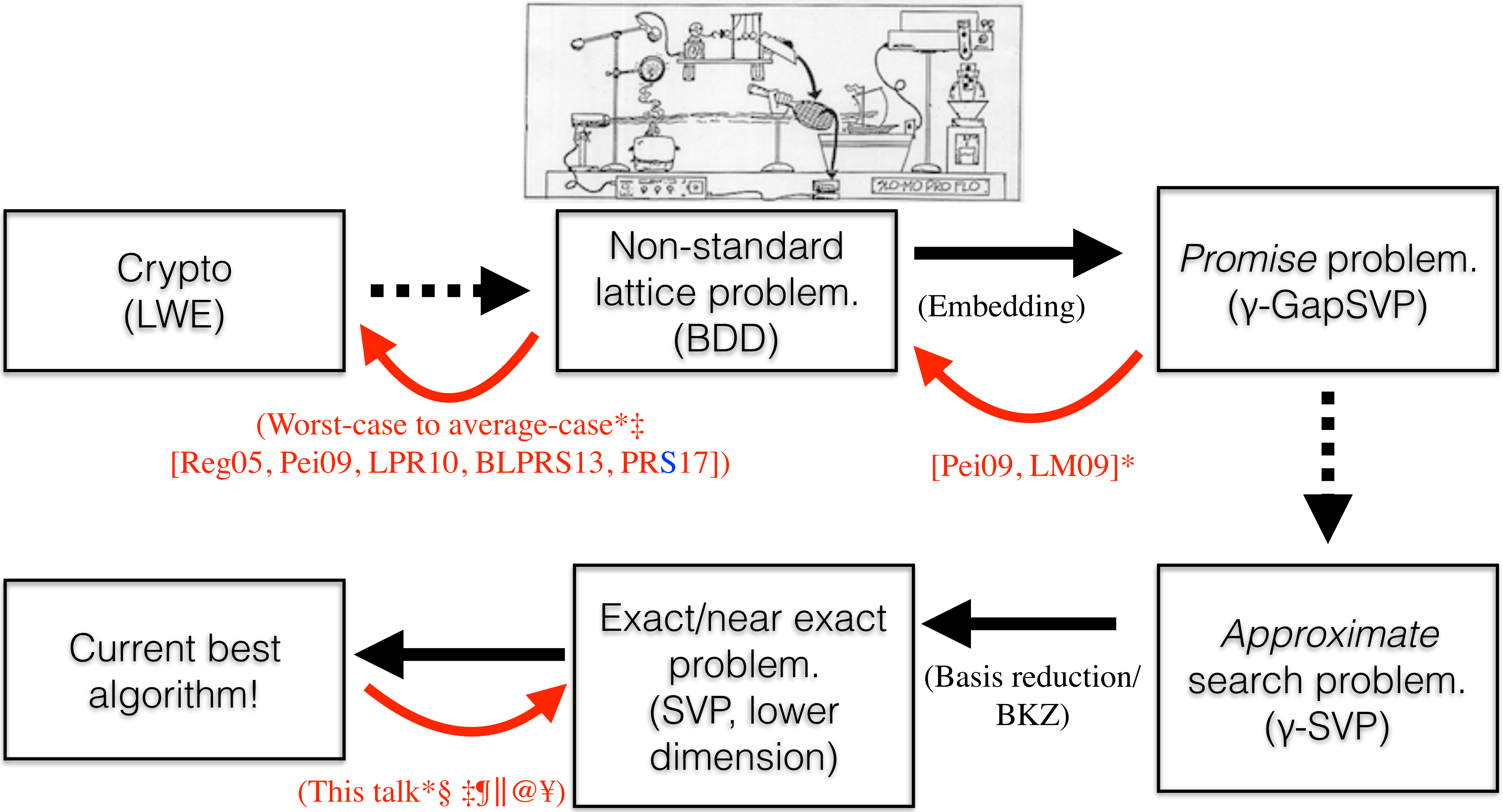

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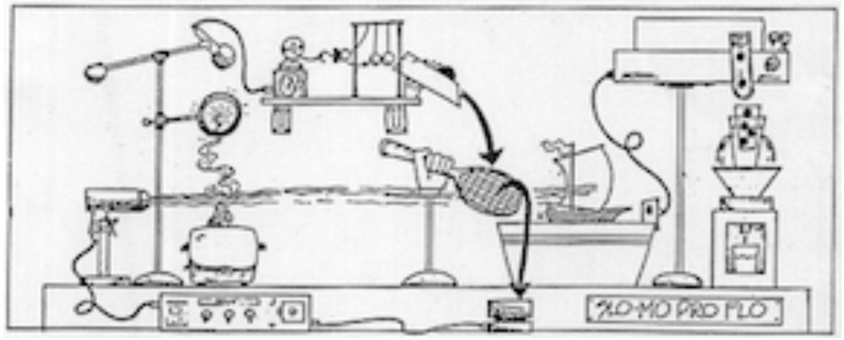

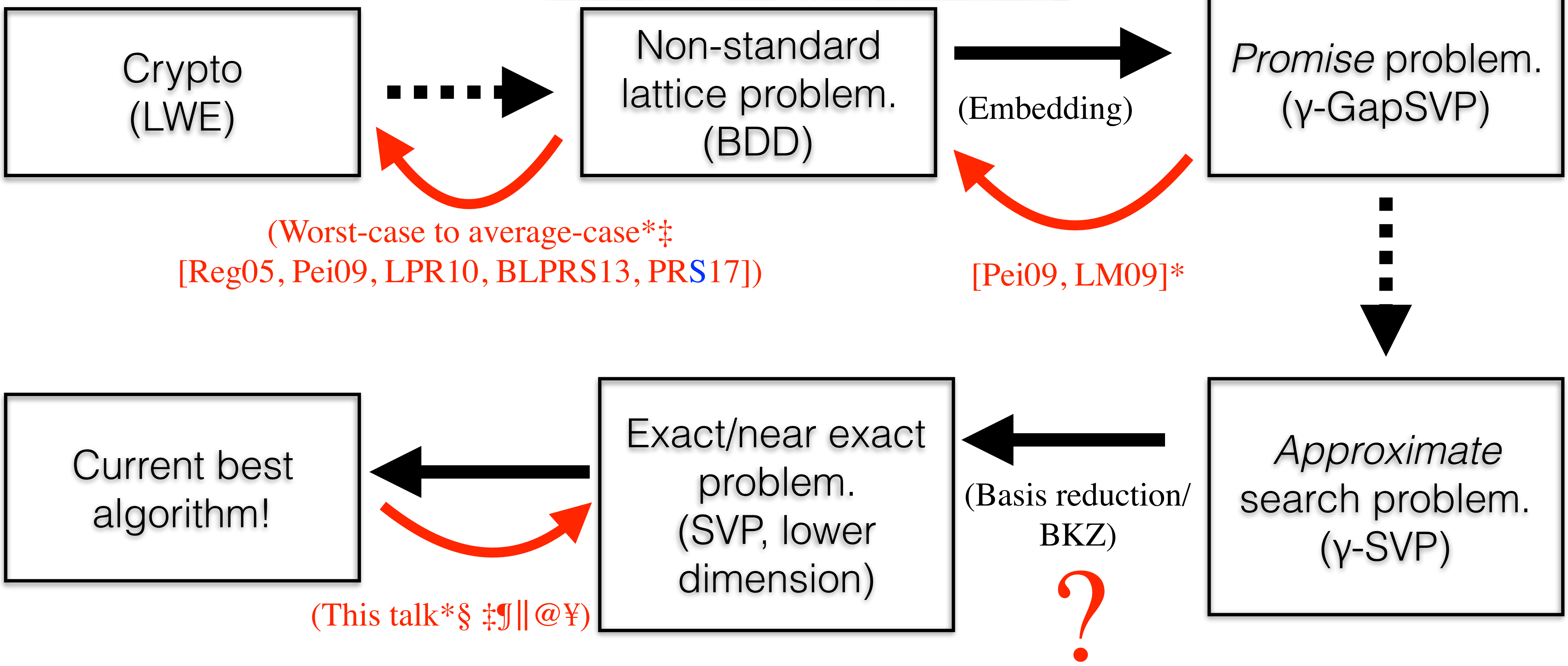

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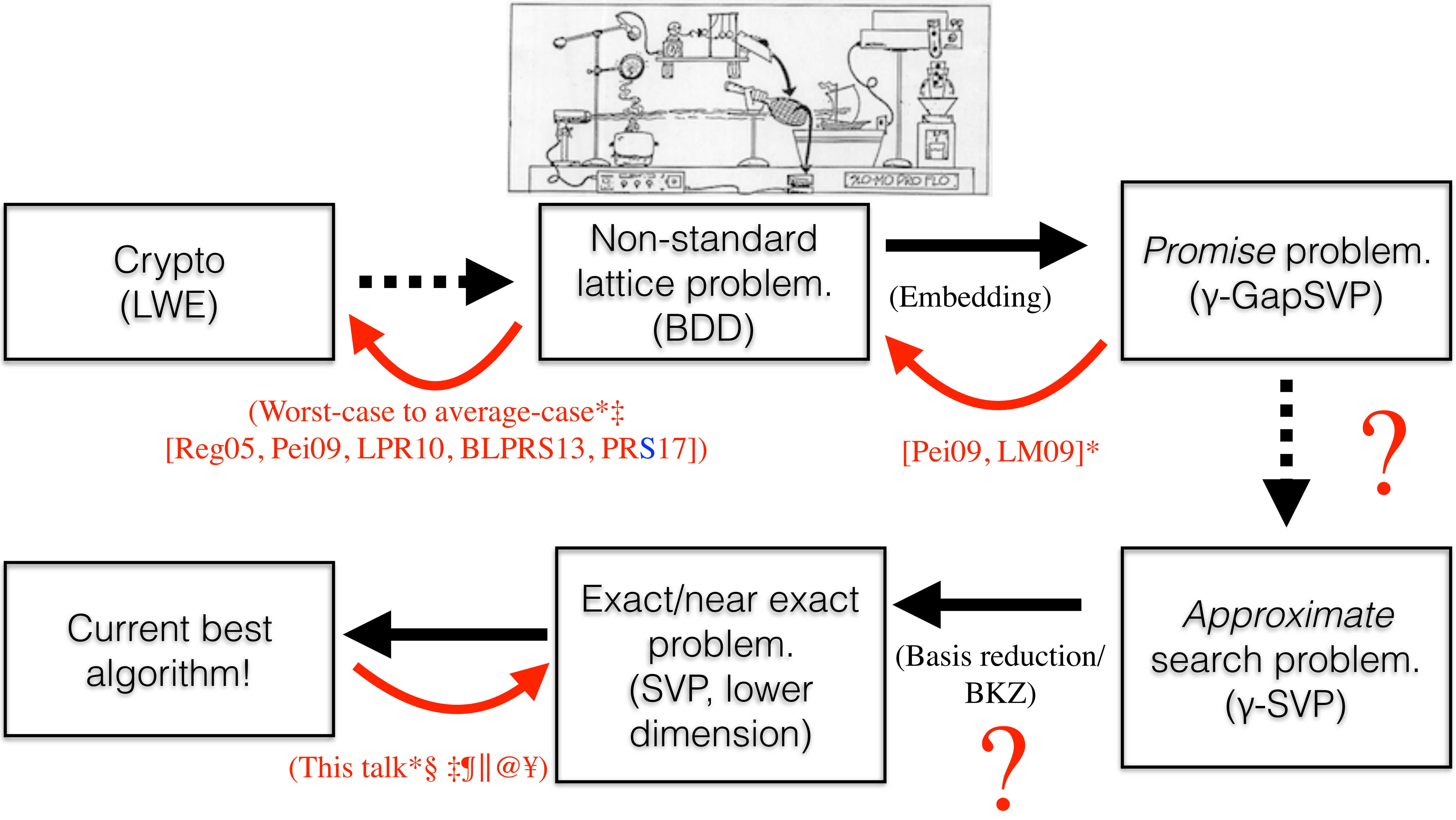

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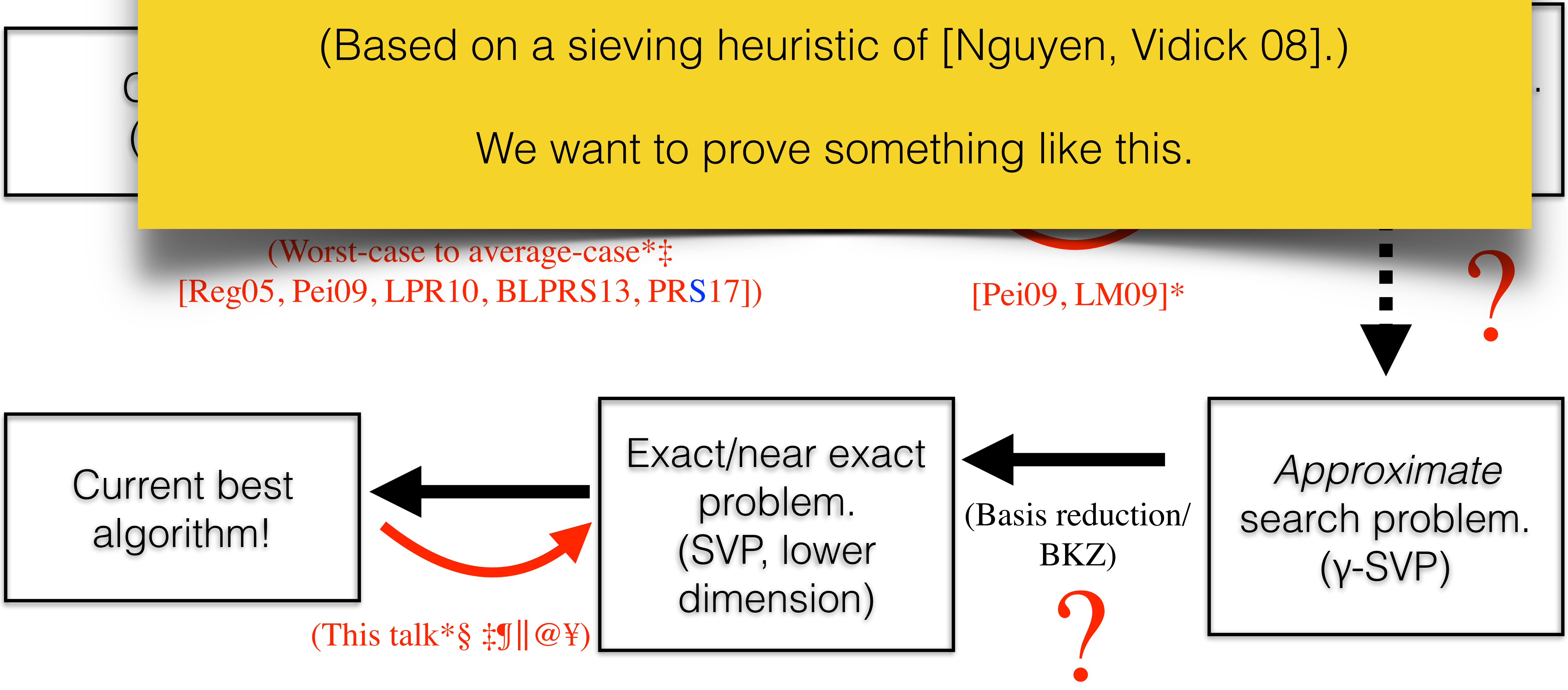
# Security of Lattice-Based Crypto

Quantitative security estimates are (all?) based on the assumption that the fastest algorithm for exact/near exact SVP runs in time $(4/3)^{n/2}$.

(Based on a sieving heuristic of [Nguyen, Vidick 08].)

We want to prove something like this.



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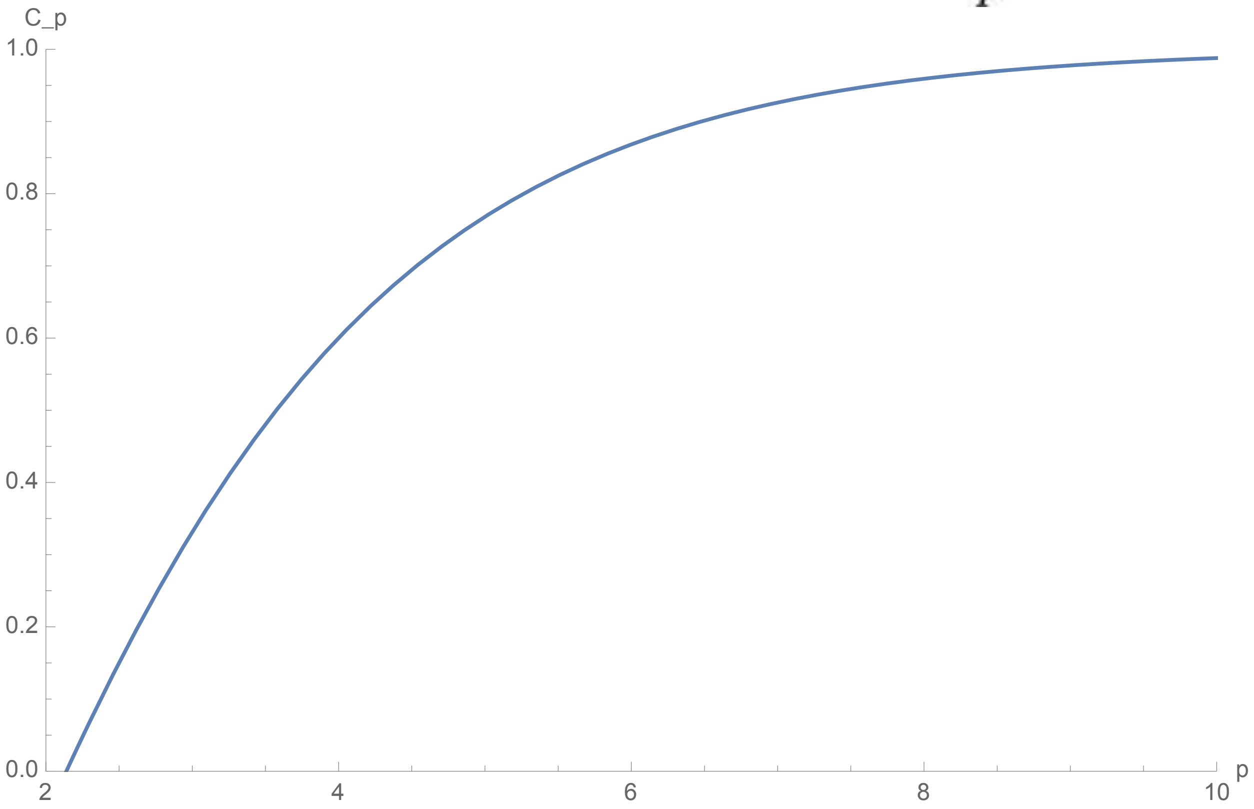

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  - Compare with the $(4/3)^{n/2}$ heuristic lower bound.

# Act 2:
# Fine-Grained Hardness of CVP

Huck
Bennett

Alexander
Golovnev

# Lattices

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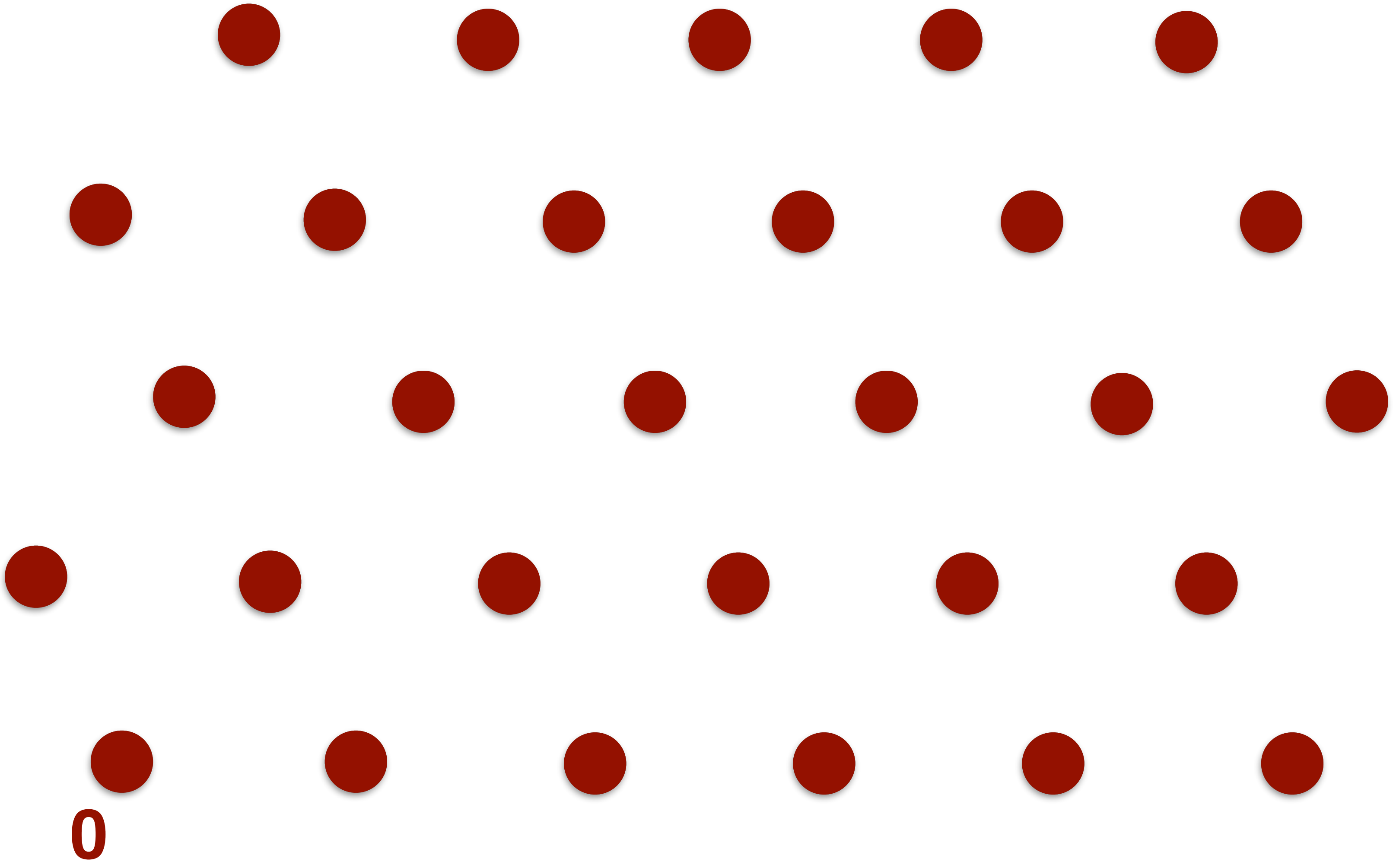

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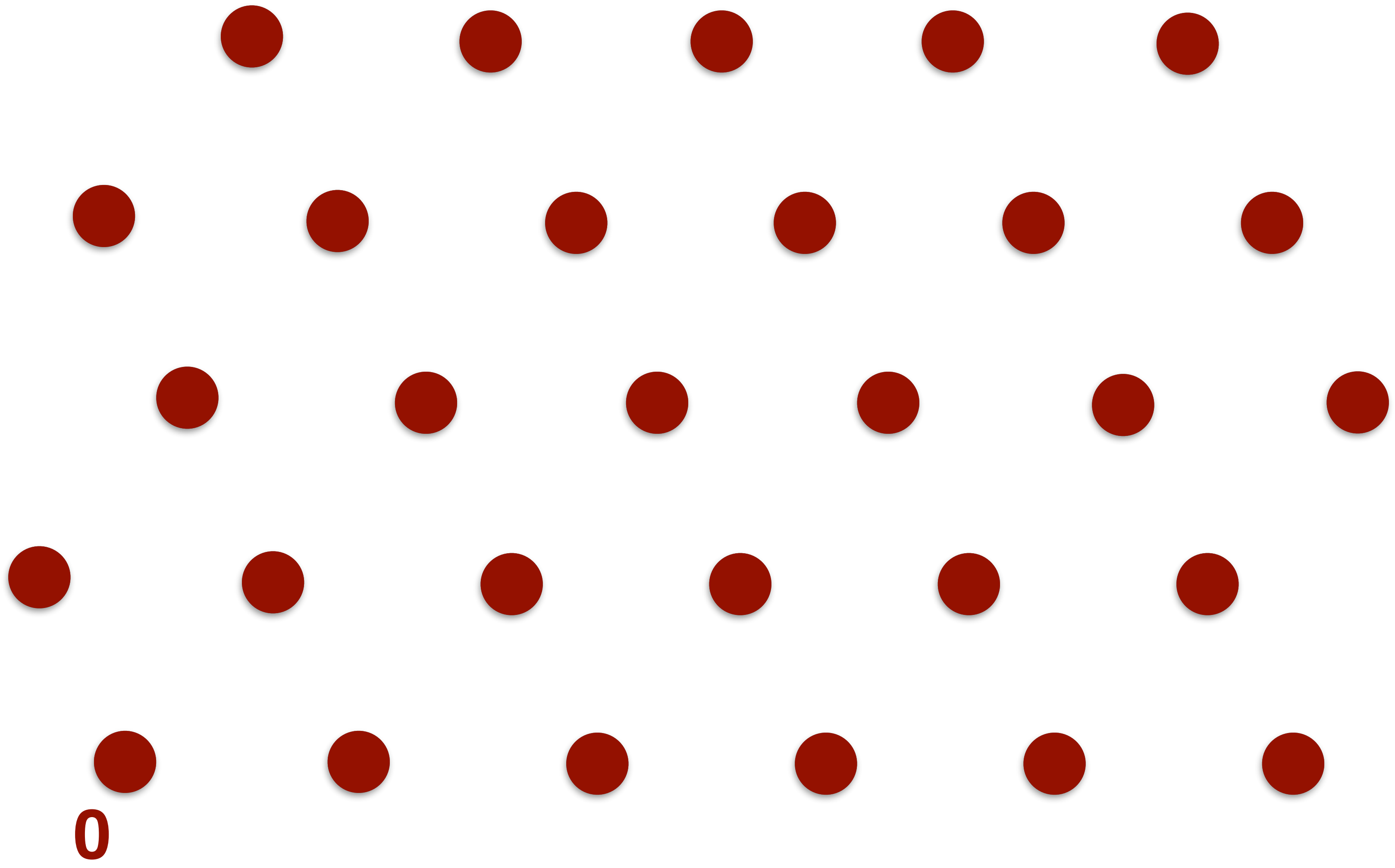

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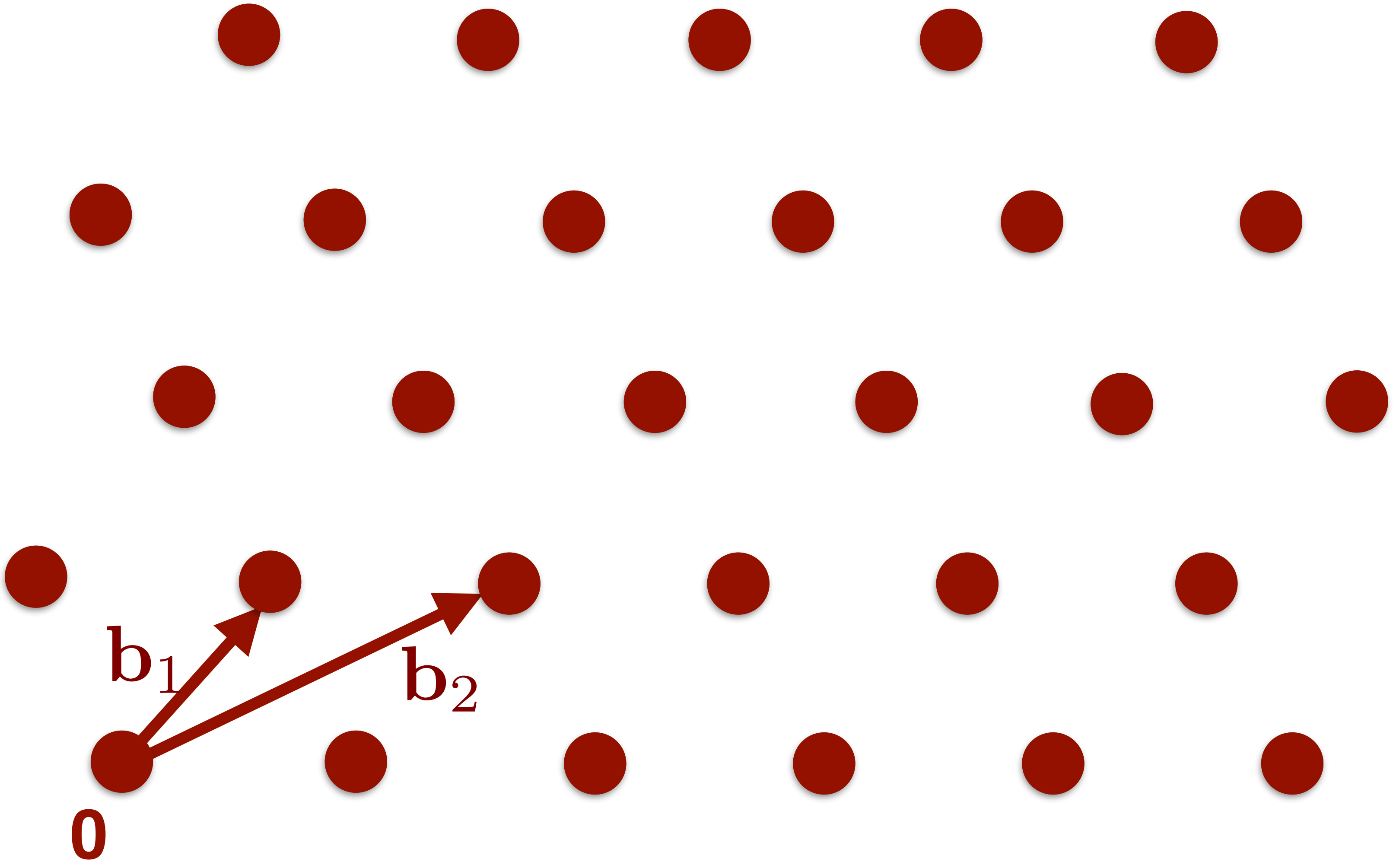

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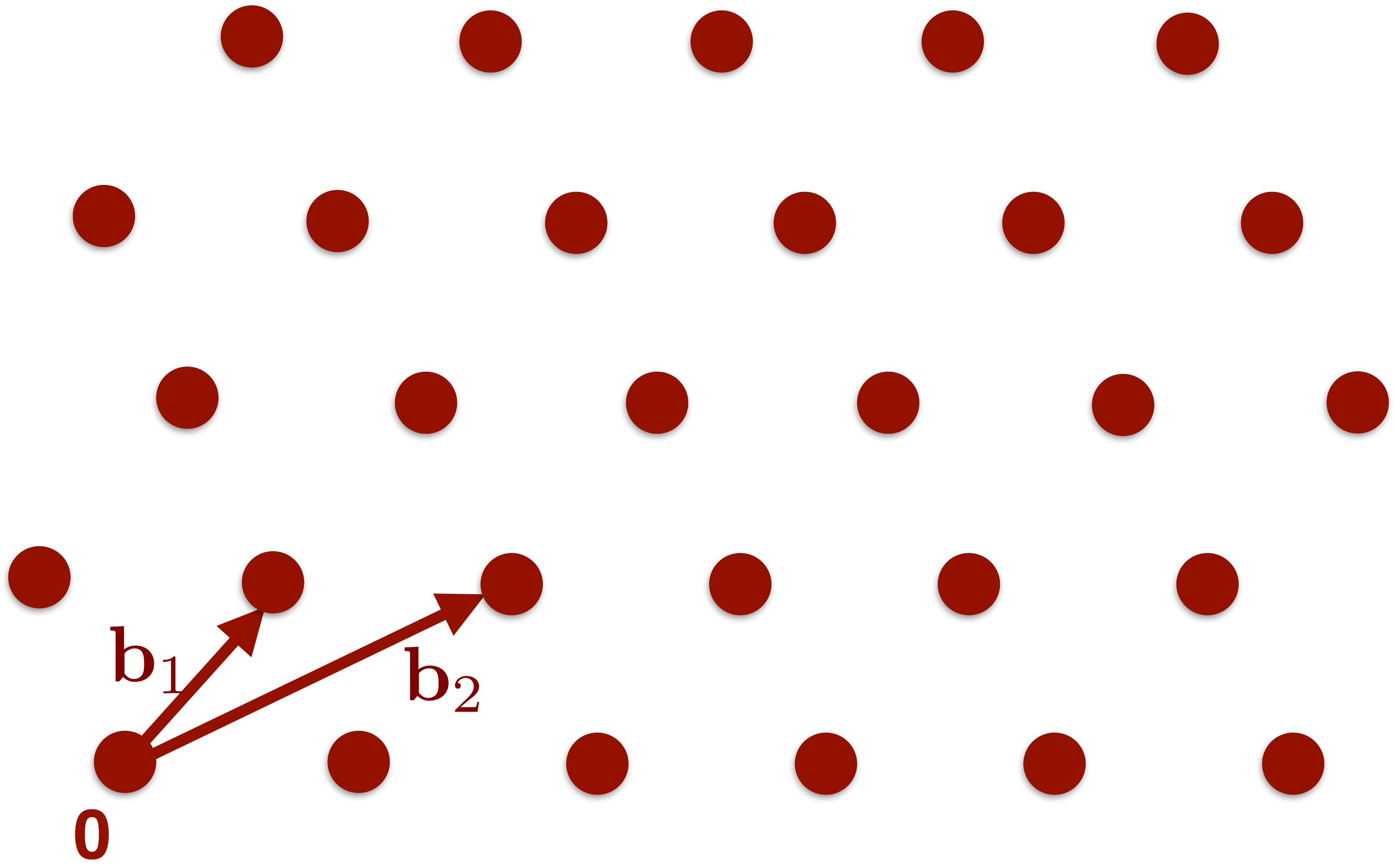

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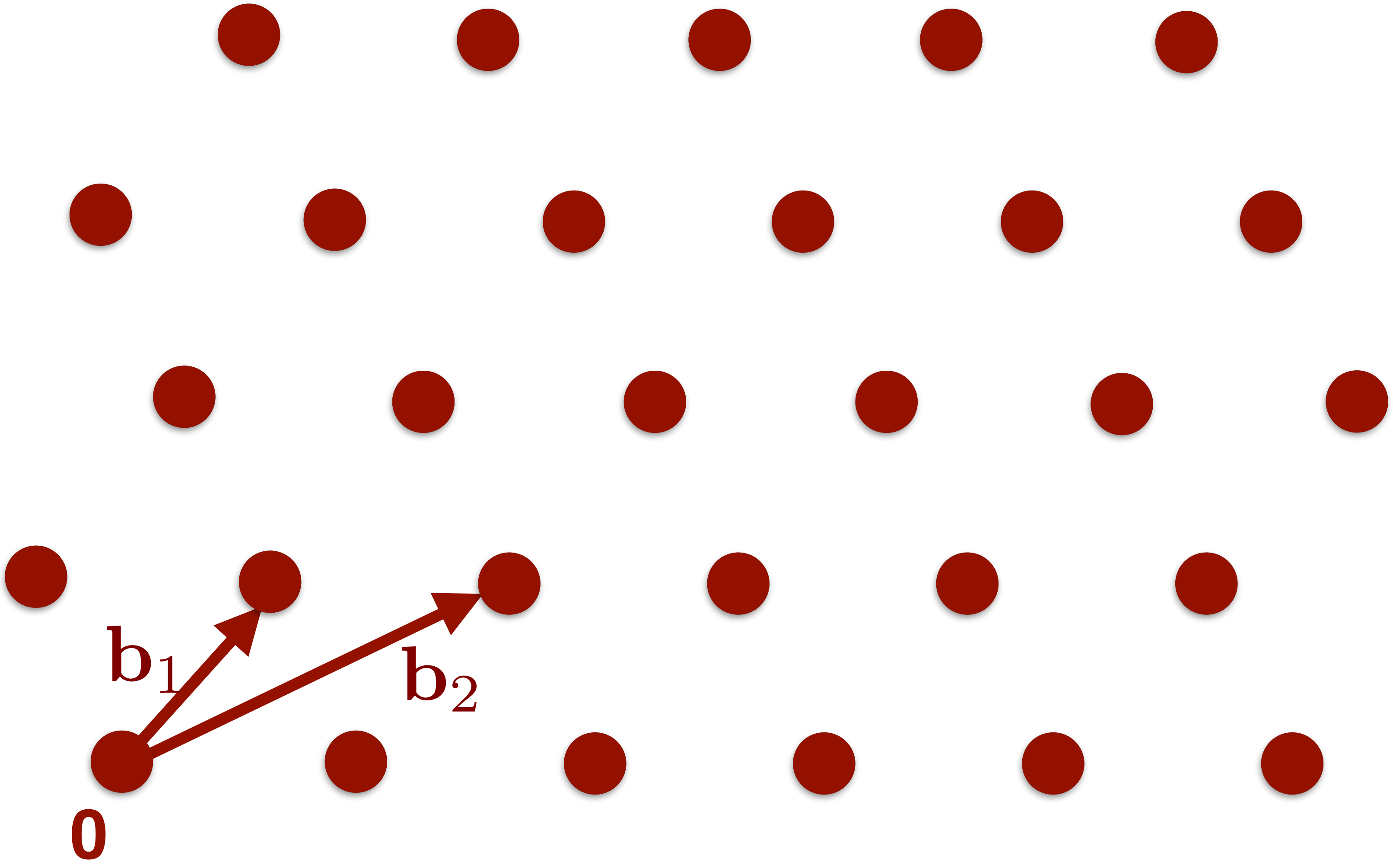
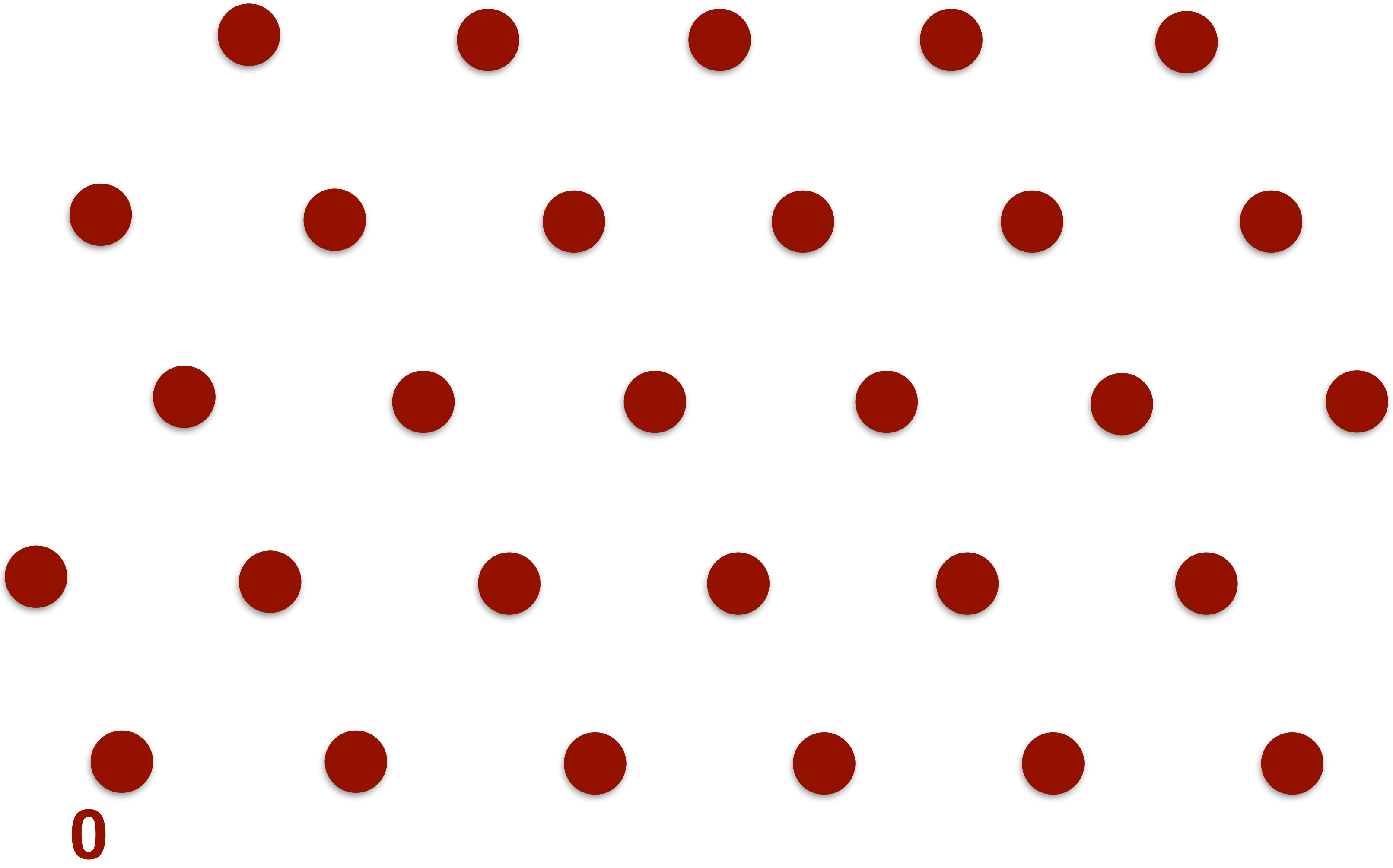
- $n$ is the *rank* of the lattice, and $d$ is the *ambient dimension*.



# The Closest Vector Problem



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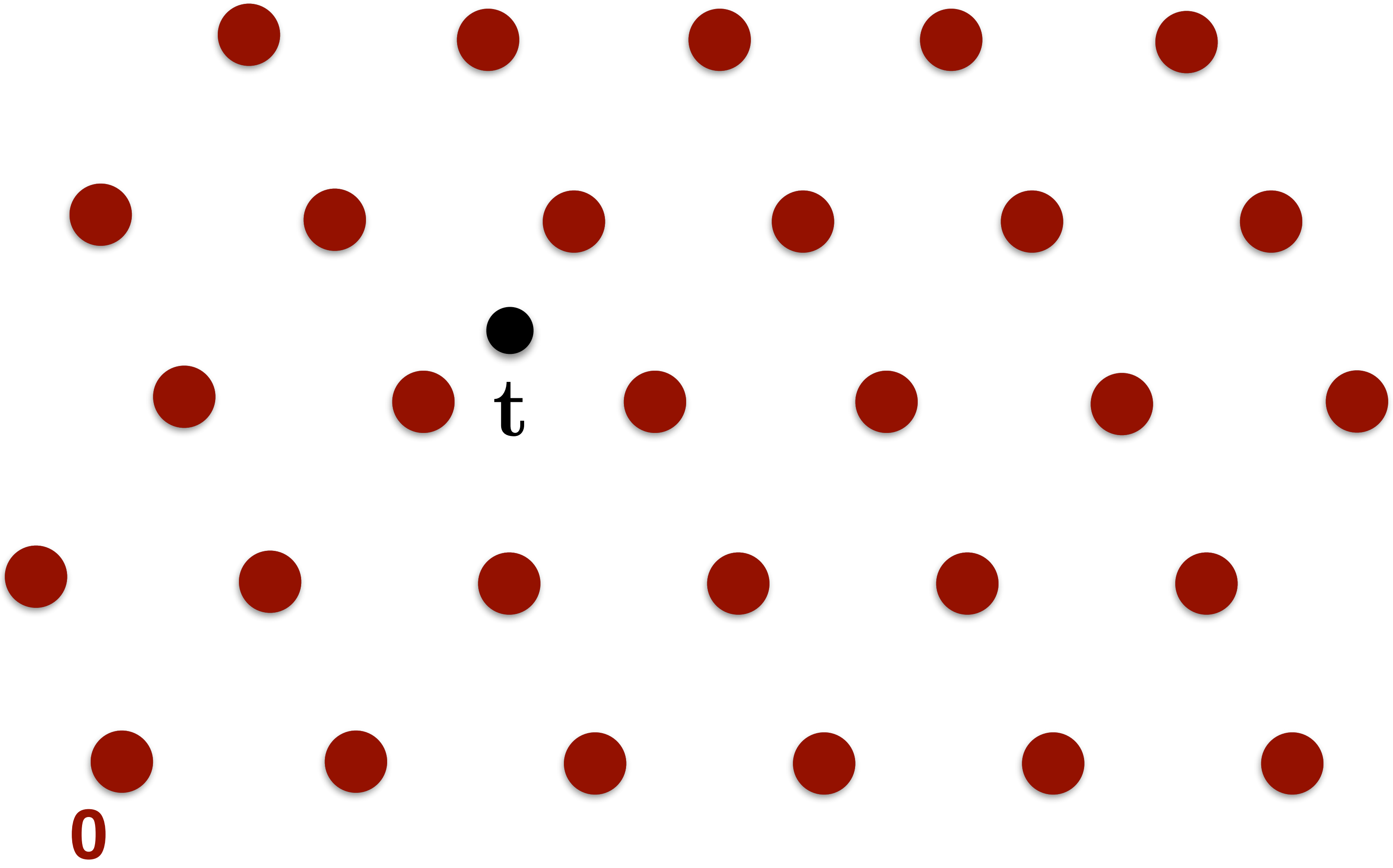

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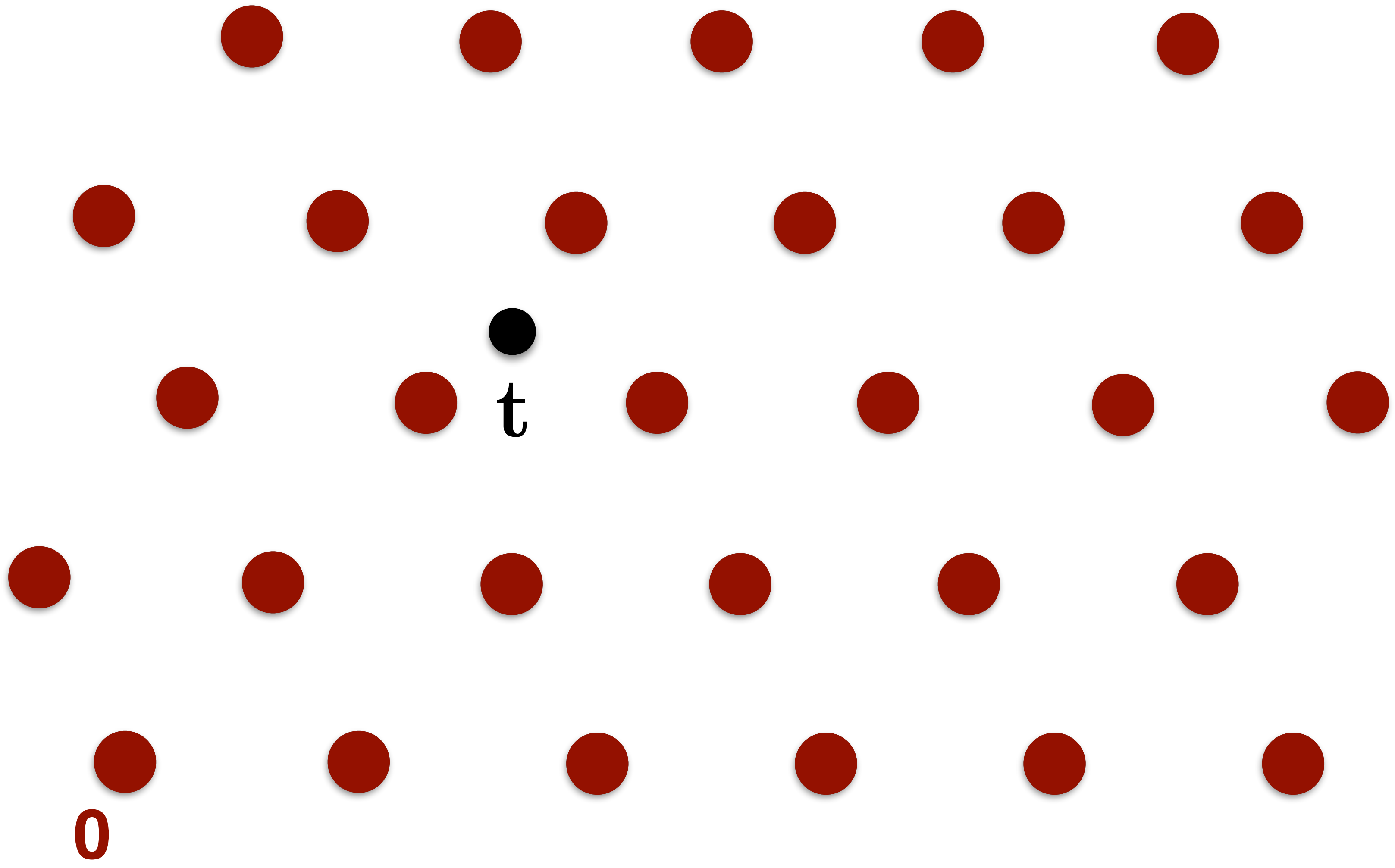
$$\mathrm{dist}_p(\mathbf{t}, \mathcal{L}) := \min_{\mathbf{y} \in \mathcal{L}} \|\mathbf{y} - \mathbf{t}\|_p$$



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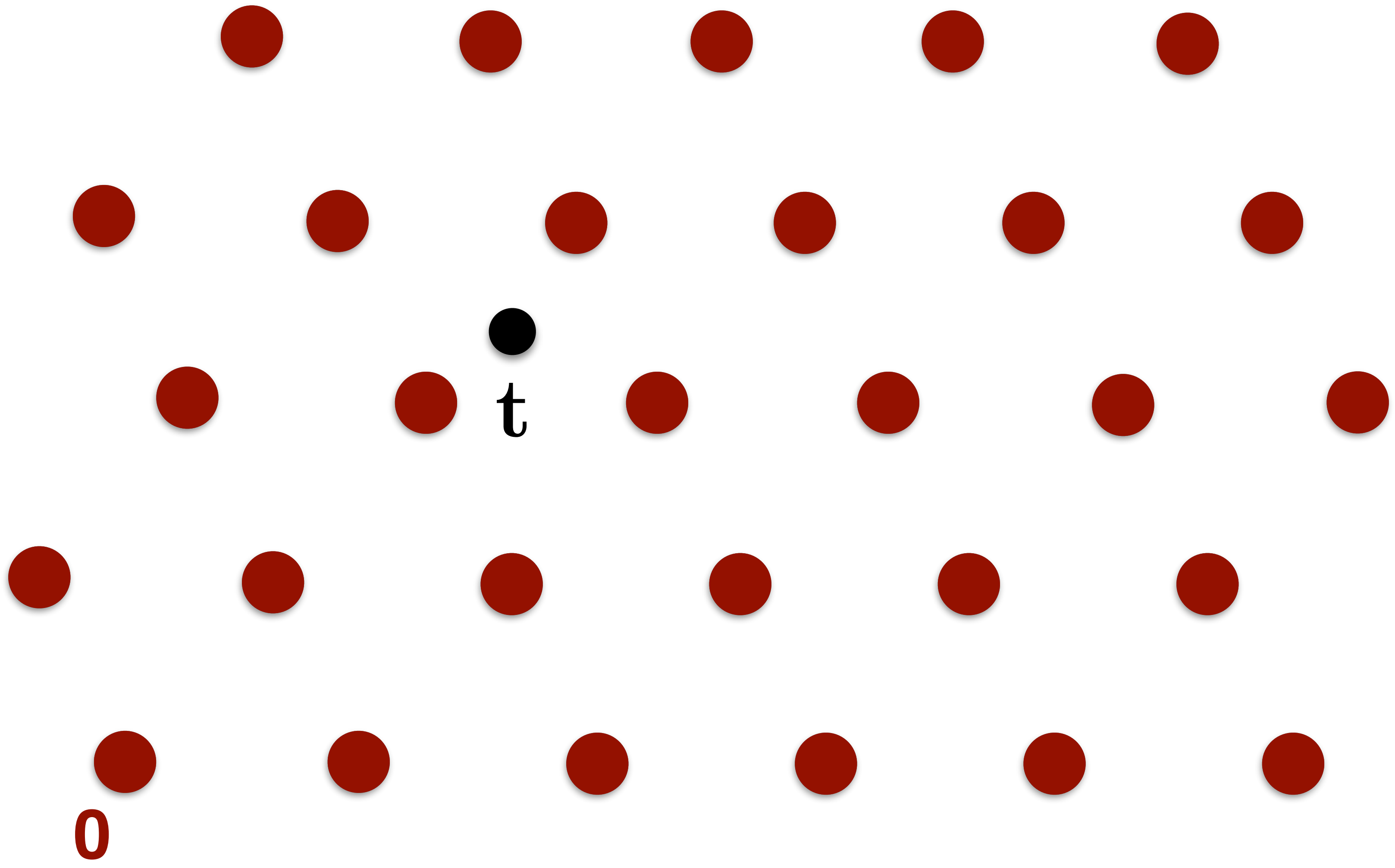

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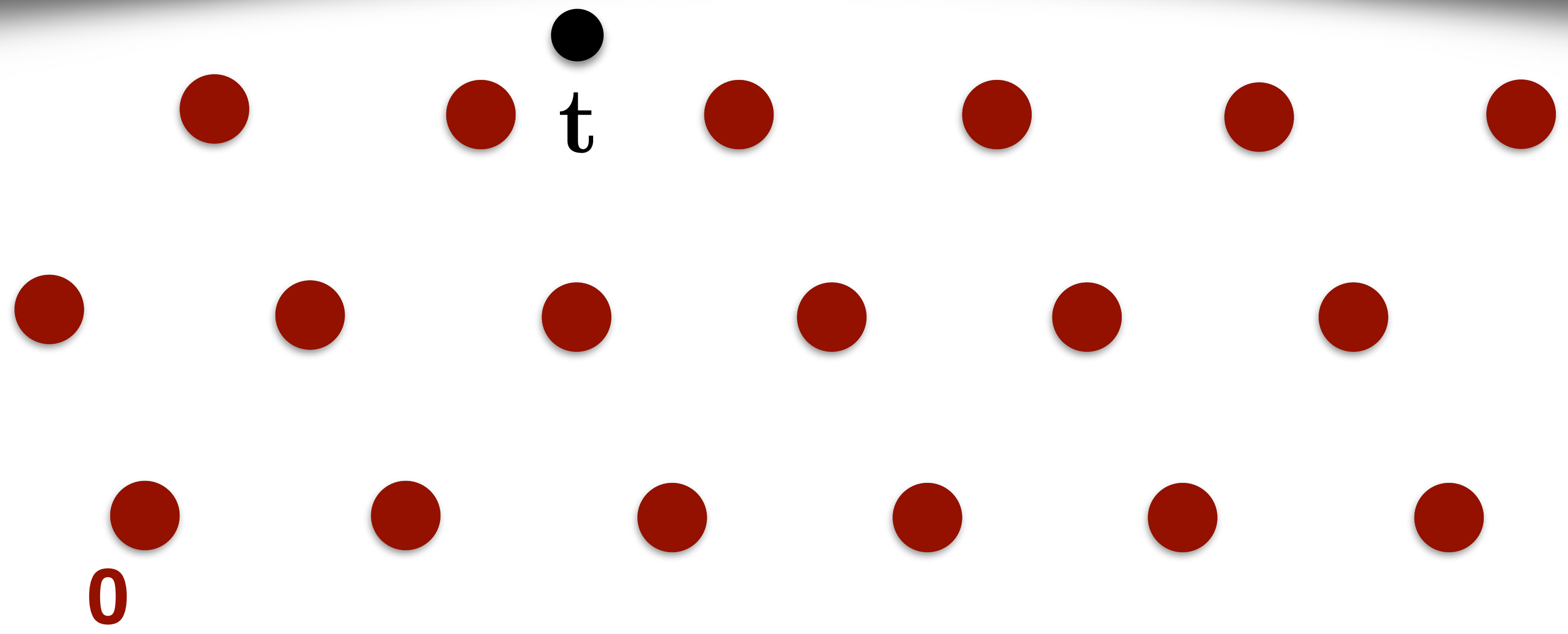

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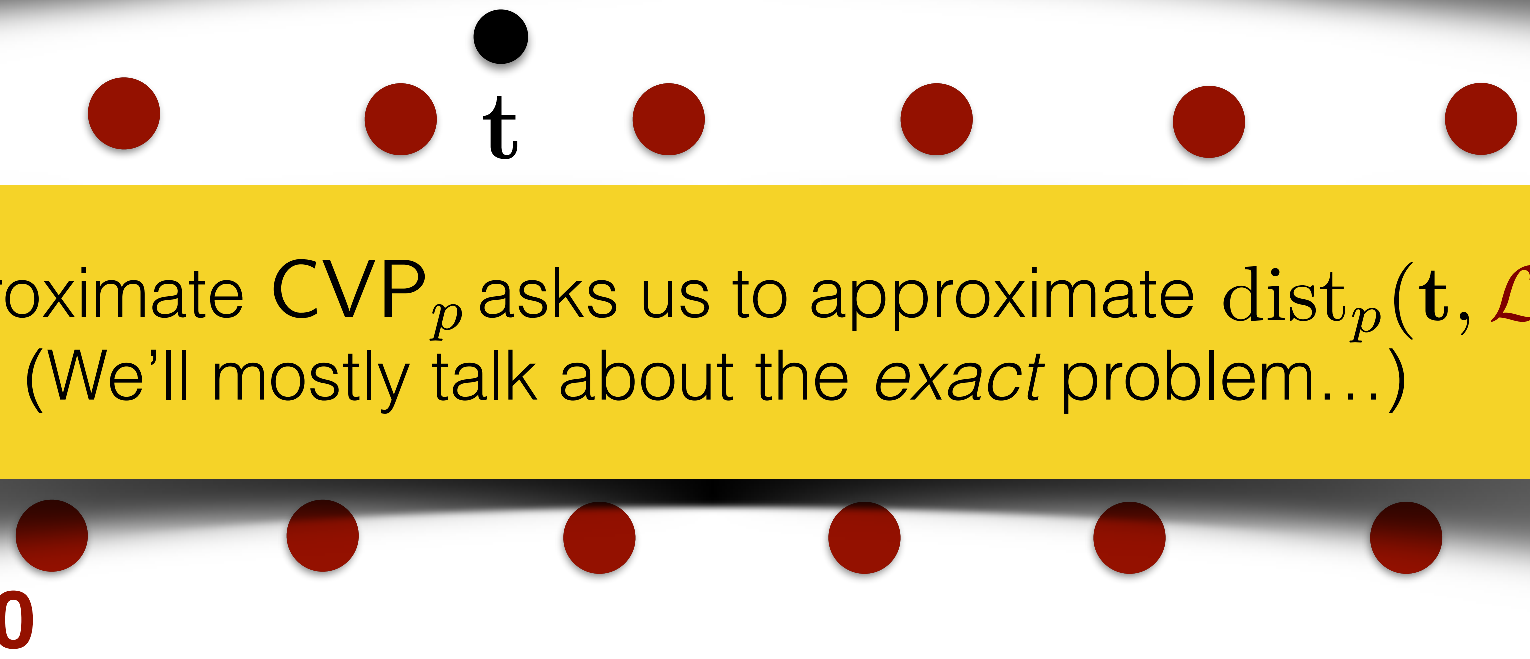

Approximate CVP$_p$ asks us to approximate $\mathrm{dist}_p(\mathbf{t}, \mathcal{L})$.
(We'll mostly talk about the *exact* problem…)

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Real-world cryptographic applications require quantitative/fine-grained hardness.

Maybe there’s a $2^{\sqrt{n}}$ -time algorithm for CVP? Even a $2^{n/20}$-time would break crypto in practice.

We’ll show something close to $2^n$-time hardness of CVP.

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Before this work, only fine-grained hardness result known was a $2^{\Omega(n)}$-time lower bound [folklore, Yeom15].

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**Conjecture** (SETH, IP99). *There exists a constant integer $k$ such that no algorithm can solve $k$-SAT in $2^{0.99n}$ time.*

We want to show a reduction from k-SAT on n variables to CVP on a lattice of rank n.

# 2-SAT
# (a very special case…)

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We can assume that $\mathbf{z} \in \{0,1\}^n$!

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Not a very safe assumption…

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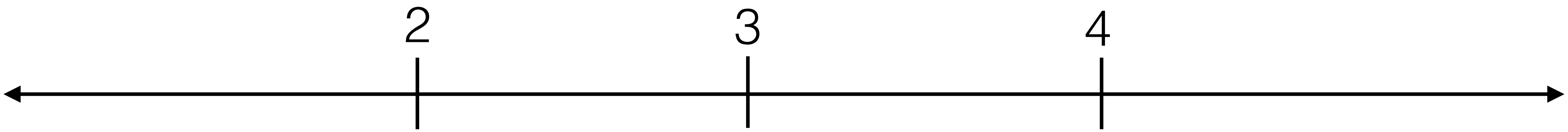

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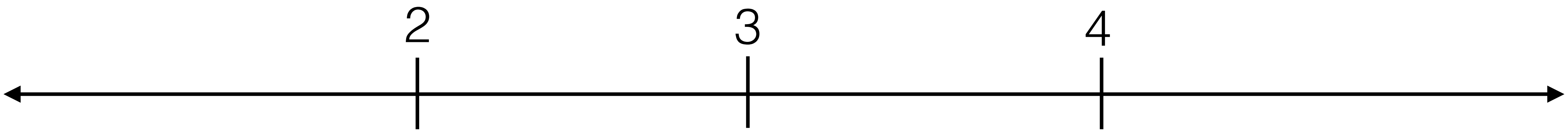

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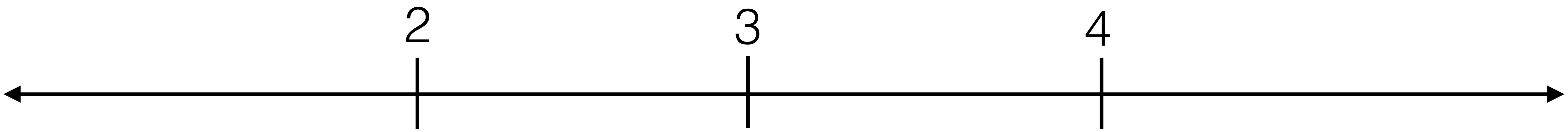

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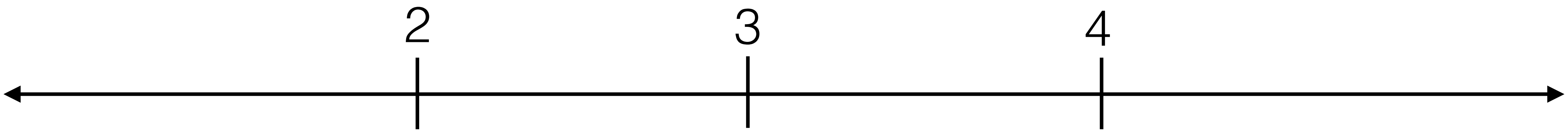
    - We can't find 3 distinct numbers that are equidistant from some other number…



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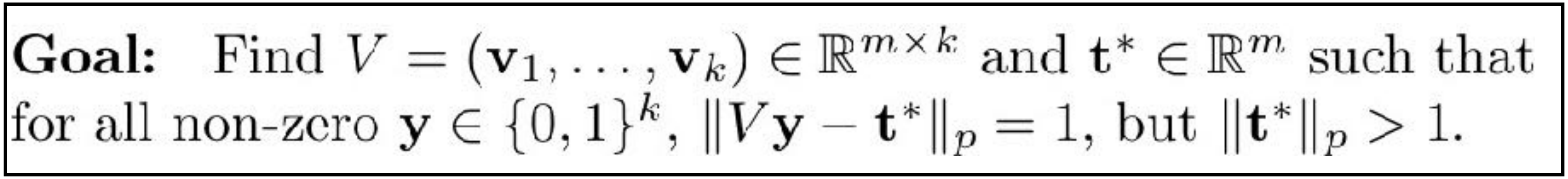

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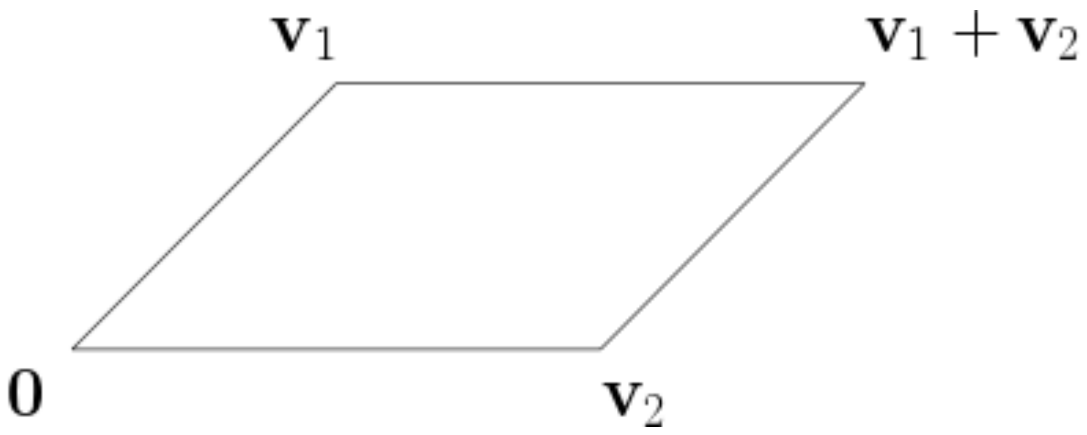

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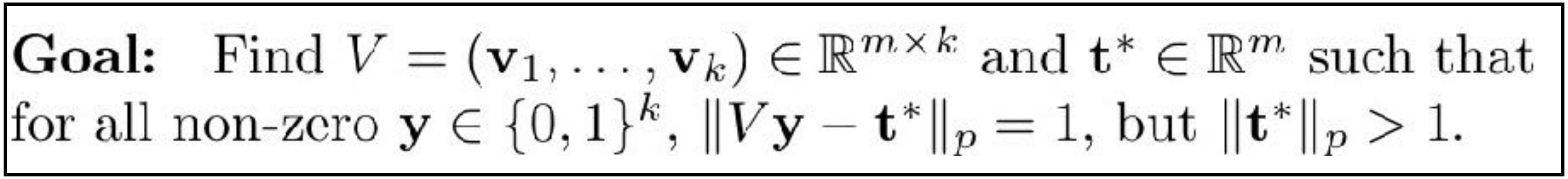

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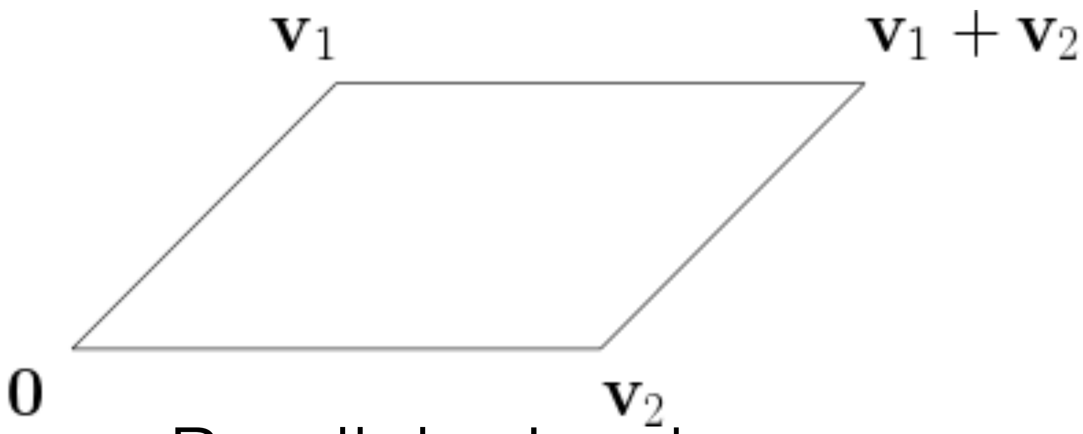

Parallelepiped

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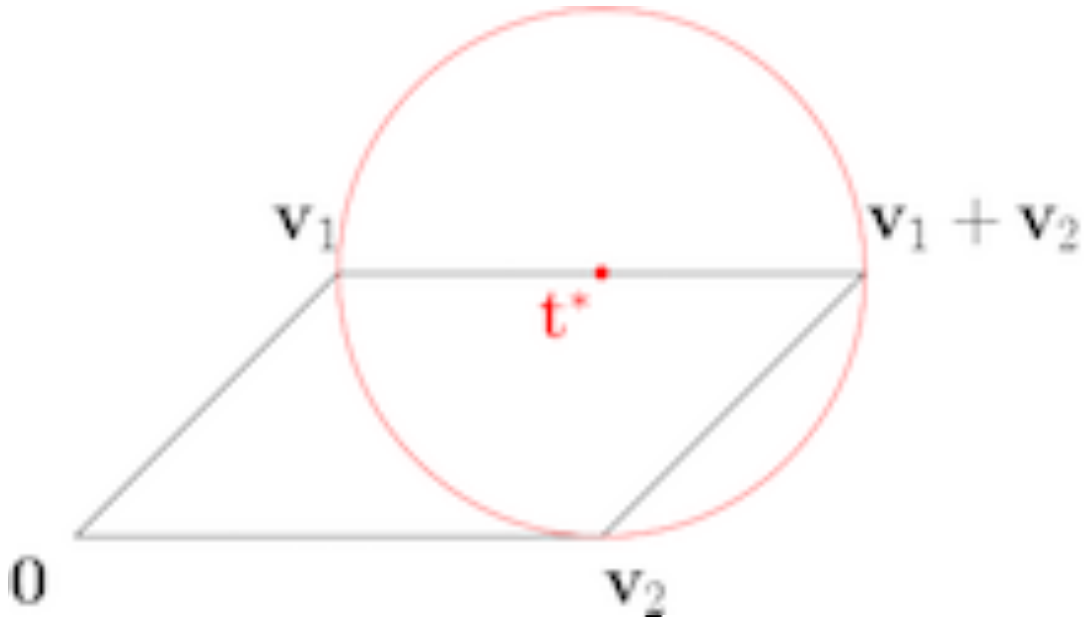

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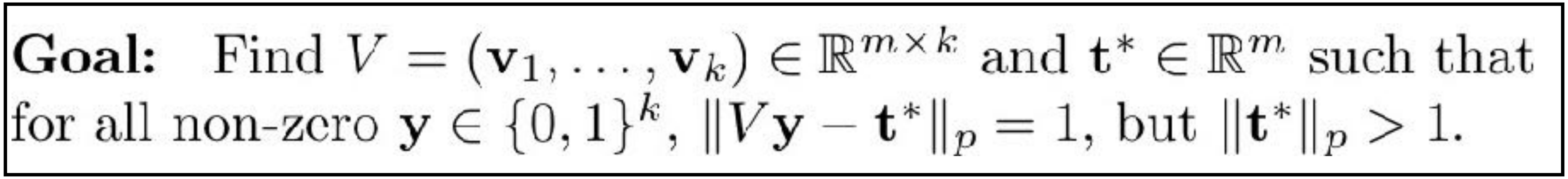

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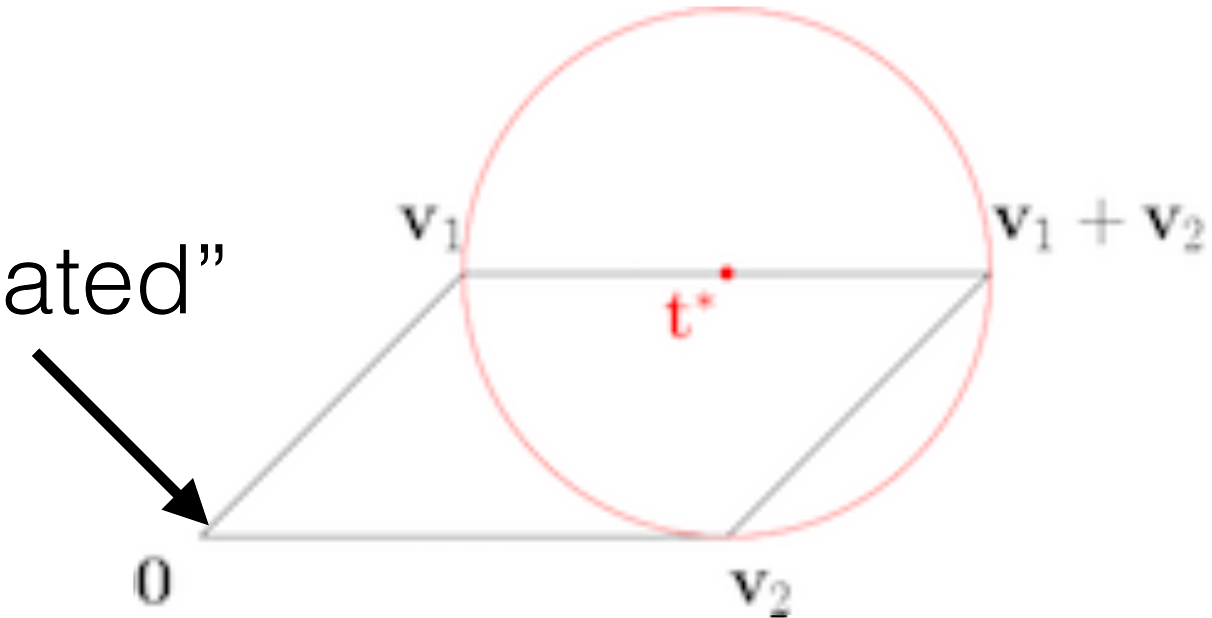

“Isolated”

Parallelepiped

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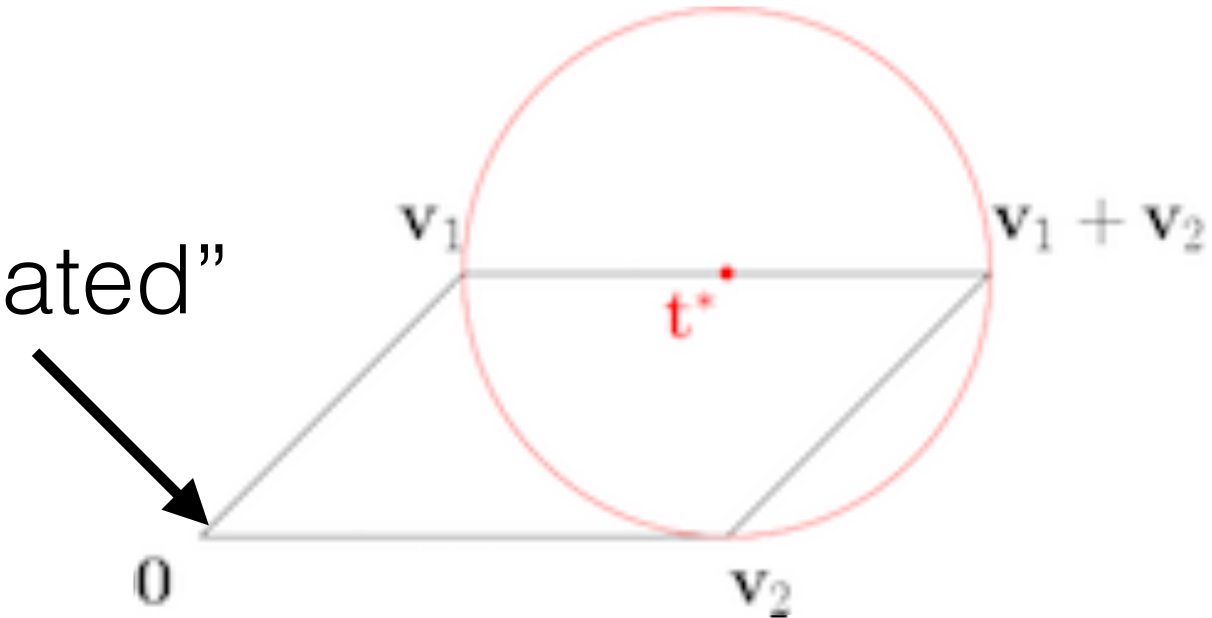

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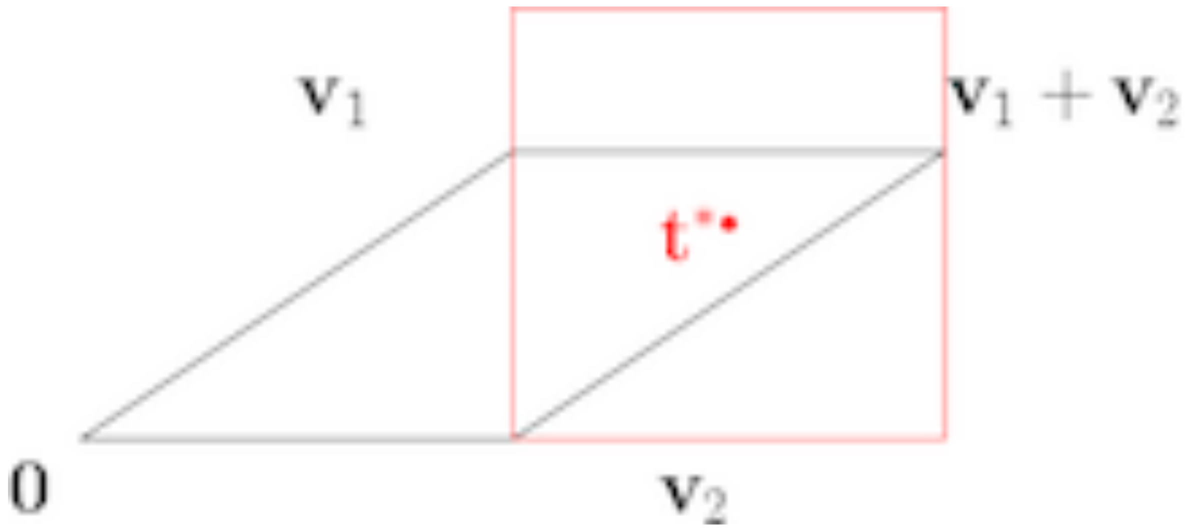

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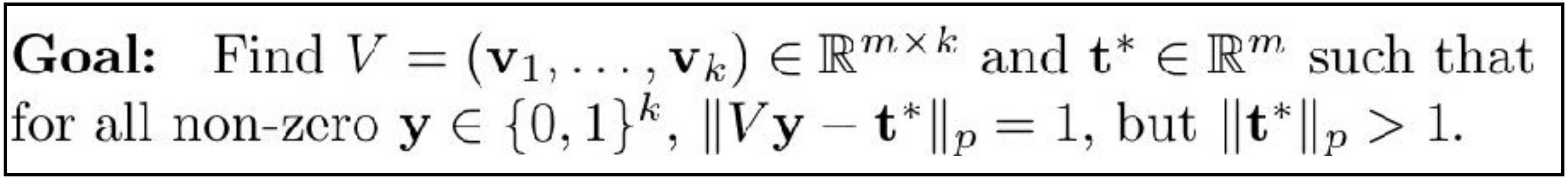

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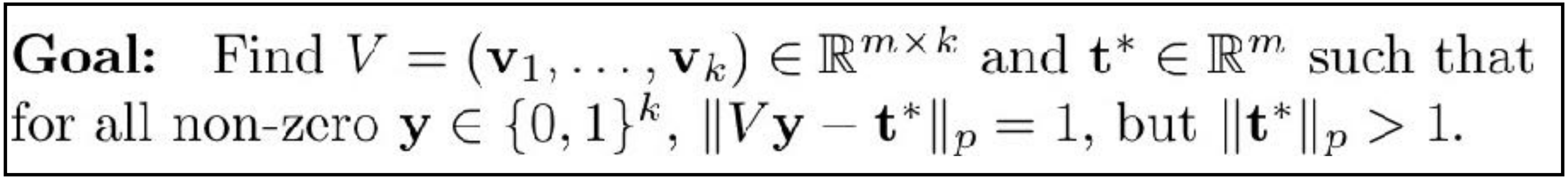

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Do these things even exist for large k?

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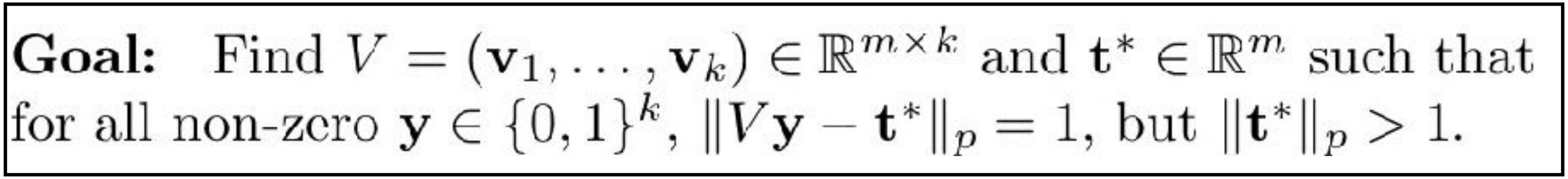

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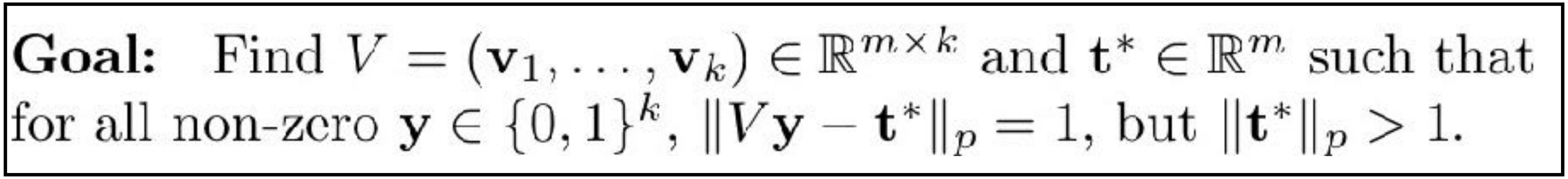

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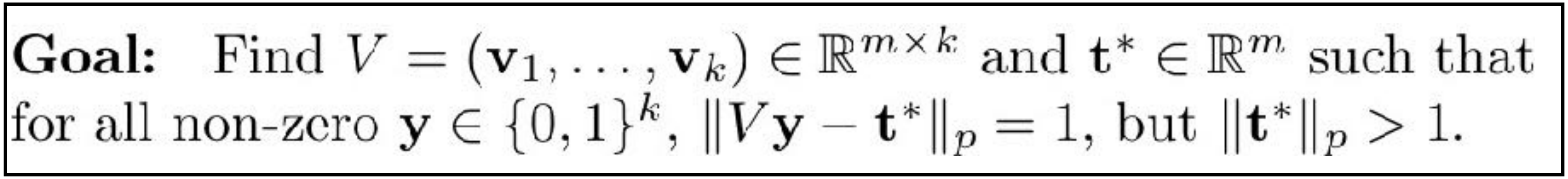

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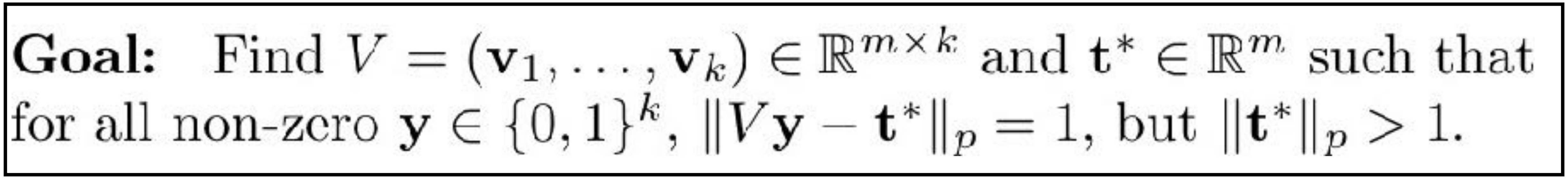

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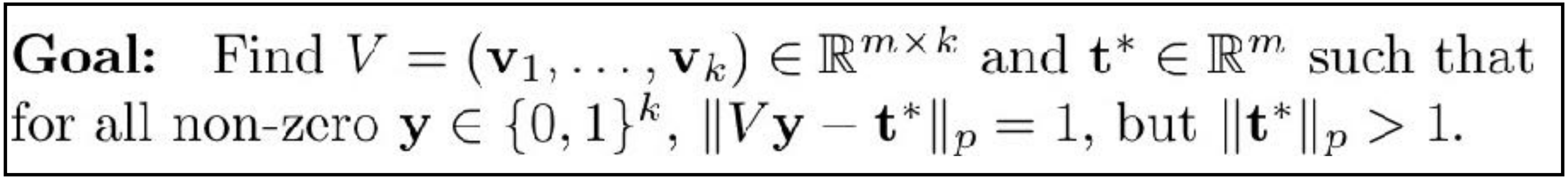

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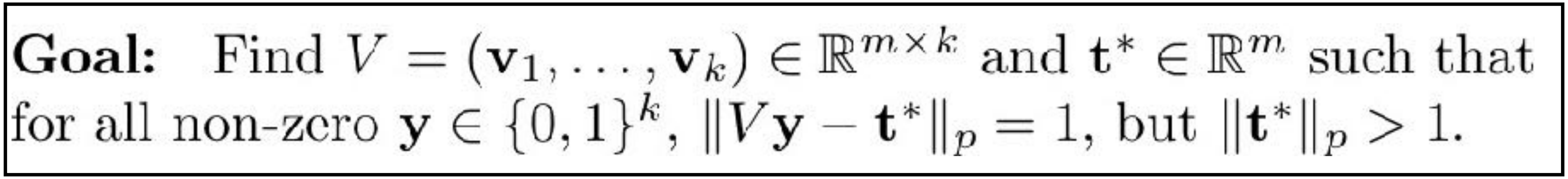

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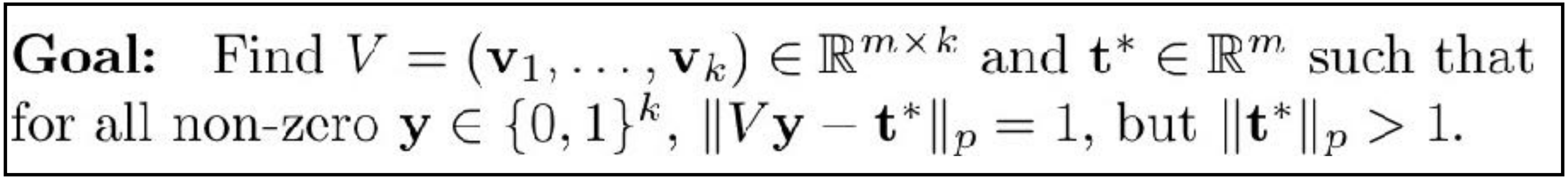

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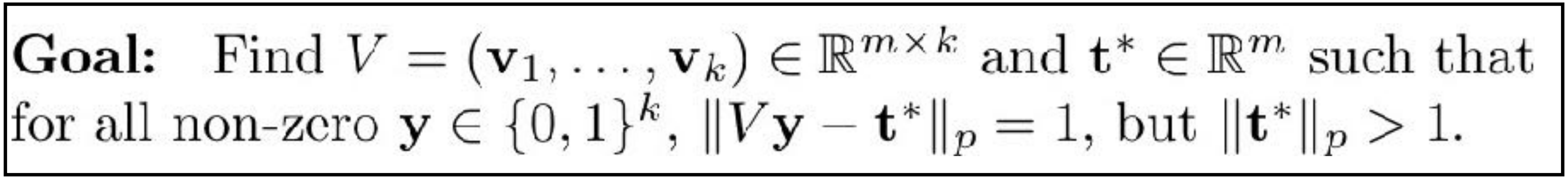

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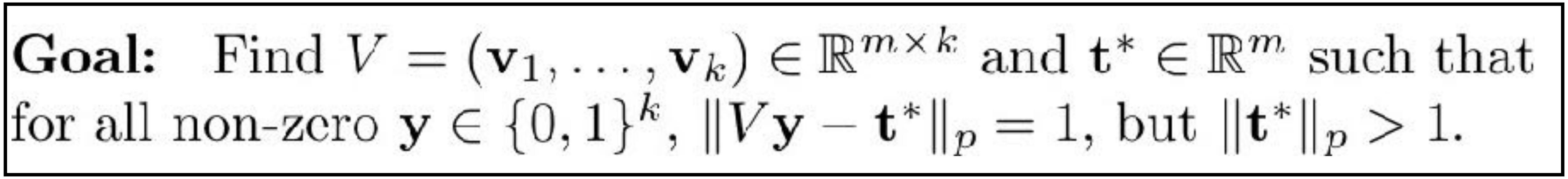

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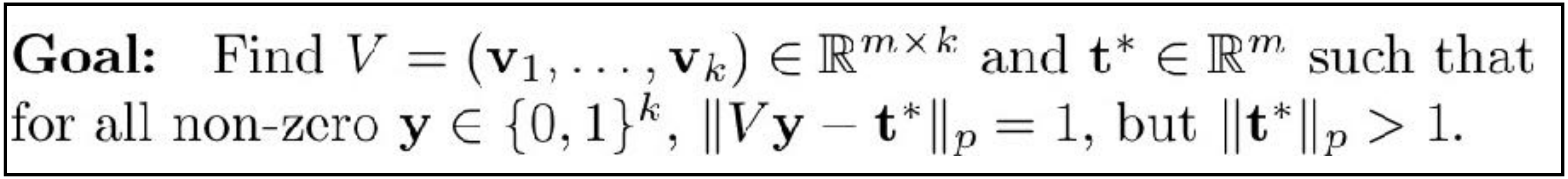

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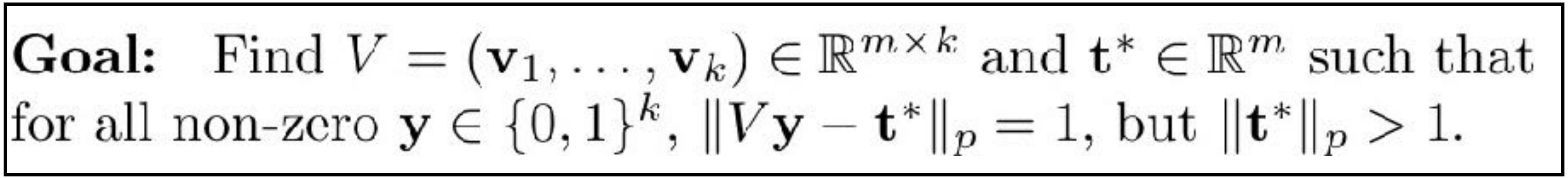

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$$\mathbf{t}^* = \begin{pmatrix} \alpha_0 t^* \\ \alpha_1 t^* \\ \alpha_1 t^* \\ \alpha_1 t^* \\ \alpha_2 t^* \\ \alpha_2 t^* \\ \alpha_2 t^* \\ \alpha_3 t^* \end{pmatrix}$$

# Isolating Parallelepipeds

$$V = \begin{array}{r} \alpha_0 \\ \alpha_1 \\ \alpha_1 \\ \alpha_1 \\ \alpha_2 \\ \alpha_2 \\ \alpha_2 \\ \alpha_3 \end{array} \!\! \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

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$$\begin{aligned} \left\| V \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \mathbf{t}^* \right\|_p^p &= |\alpha_0|^p |2 - t^*|^p \\ &+ |\alpha_1|^p \cdot (|2 - t^*|^p + 2|t^*|^p) \\ &+ |\alpha_2|^p \cdot (|-2 - t^*|^p + 2|t^*|^p) \\ &+ |\alpha_3|^p \cdot |-2 - t^*|^p \end{aligned}$$

# Isolating Parallelepipeds

$$V = \begin{array}{c|ccc|} \alpha_0 & 1 & 1 & 1 \\ \alpha_2 & -1 & -1 & 1 \\ \alpha_3 & -1 & -1 & -1 \end{array} \qquad \begin{pmatrix} \alpha_0 t^* \\ \alpha_2 t^* \\ \alpha_3 t^* \end{pmatrix}$$

This is linear in the $|\alpha_i|^p$!

So, it suffices to find $t^*$ such that the resulting system of linear equations in the $|\alpha_i|^p$ has a (non-negative) solution.

$$\left\| V \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \mathbf{t}^* \right\|_p^p = |\alpha_0|^p |2 - t^*|^p$$

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Want to solve $M(t^*) \cdot \begin{pmatrix} |\alpha_0|^p \\ |\alpha_1|^p \\ \vdots \\ |\alpha_k|^p \end{pmatrix} = \begin{pmatrix} 1+\varepsilon \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

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  - Piecewise polynomial in $t^*$ when p is an integer.
  - $\det(M(t^*))$ is a piecewise polynomial in $t^*$ .
  - We show that it is not always the zero polynomial when p is odd.

# Also, Approximate CVP

Max-2-SAT reduction ⇒ hardness of $(1+\varepsilon)$-approx $\mathsf{CVP}_p$ for all $p$.

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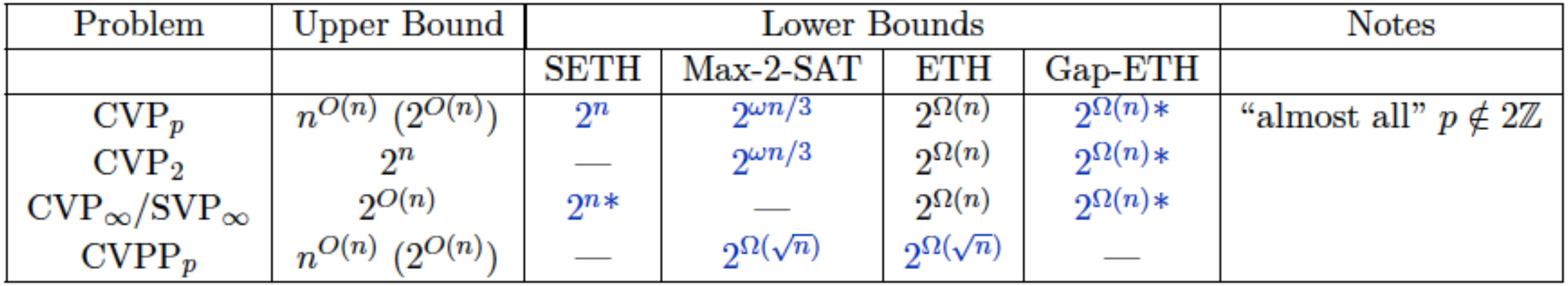
Formally GapETH hardness.

No $2^{o(n)}$-time for approx Max-2-SAT $\Rightarrow$ No $2^{o(n)}$-time for approx $\mathsf{CVP}_p$.

# Summary of Results



| Problem | Upper Bound | Lower Bounds | | | | Notes |
|---|---|---|---|---|---|---|
| | | SETH | Max-2-SAT | ETH | Gap-ETH | |
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Blue = new result.
(…) = approximation algorithm
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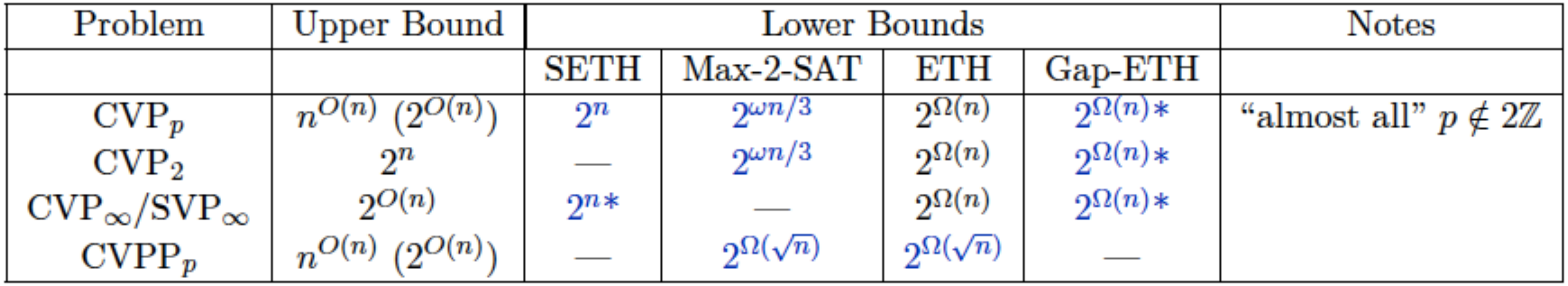

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**Pros**

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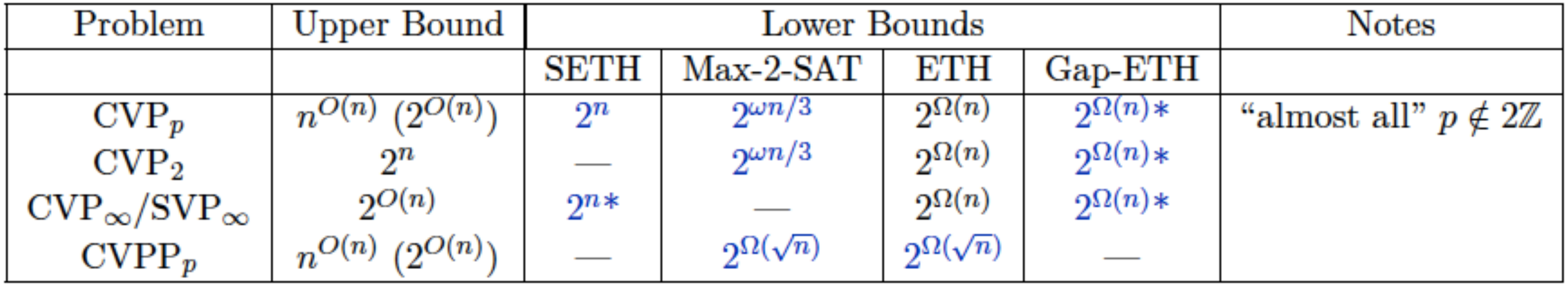

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**<u>Pros</u>**

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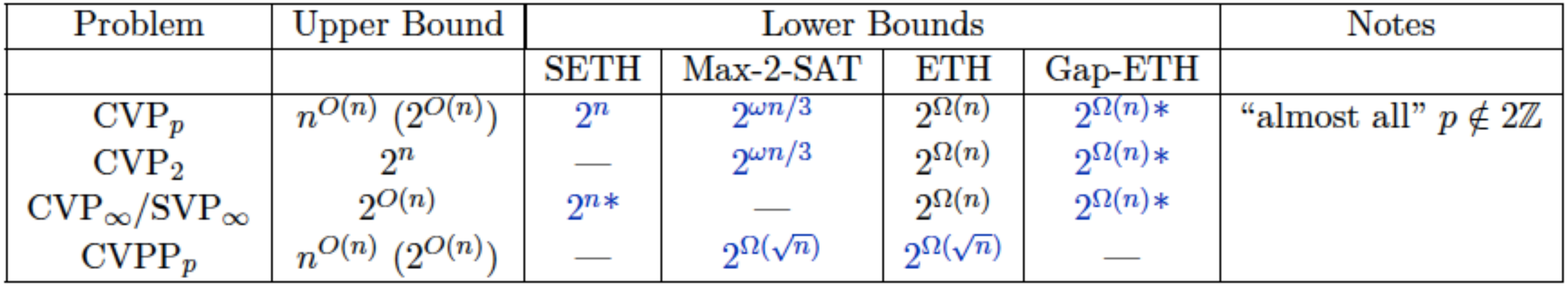

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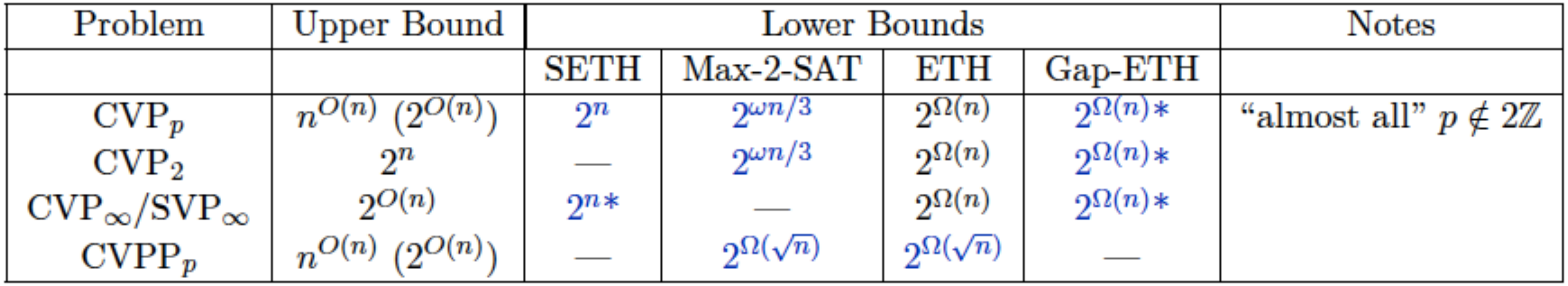

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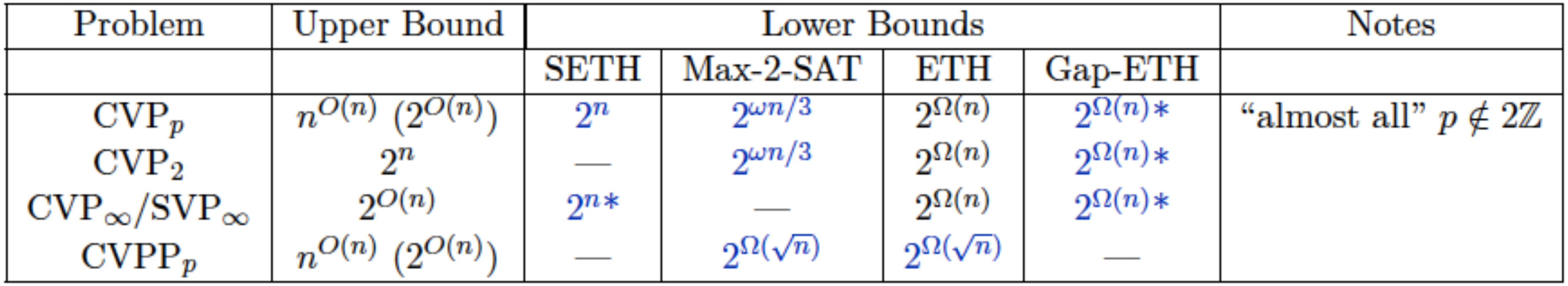

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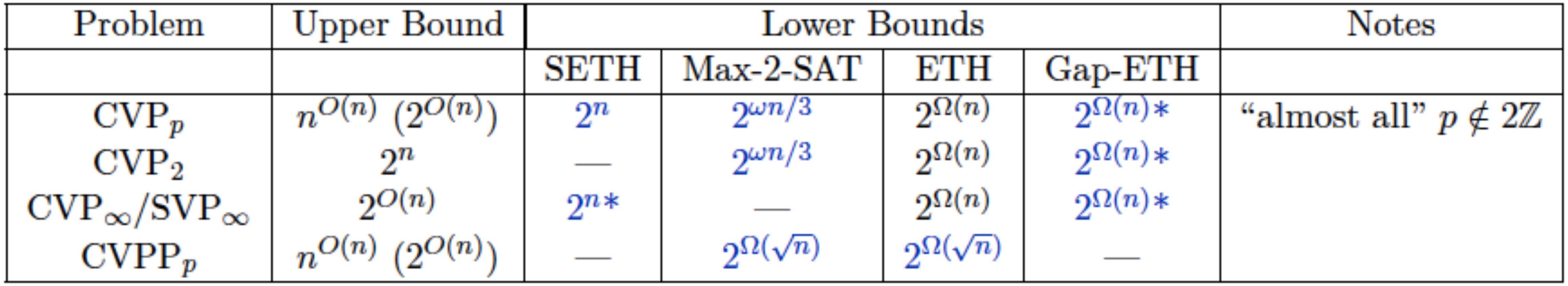

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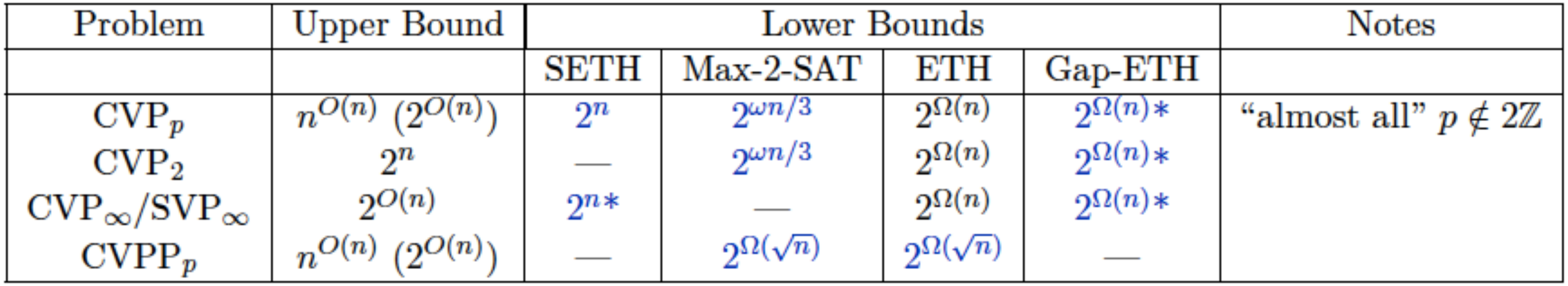

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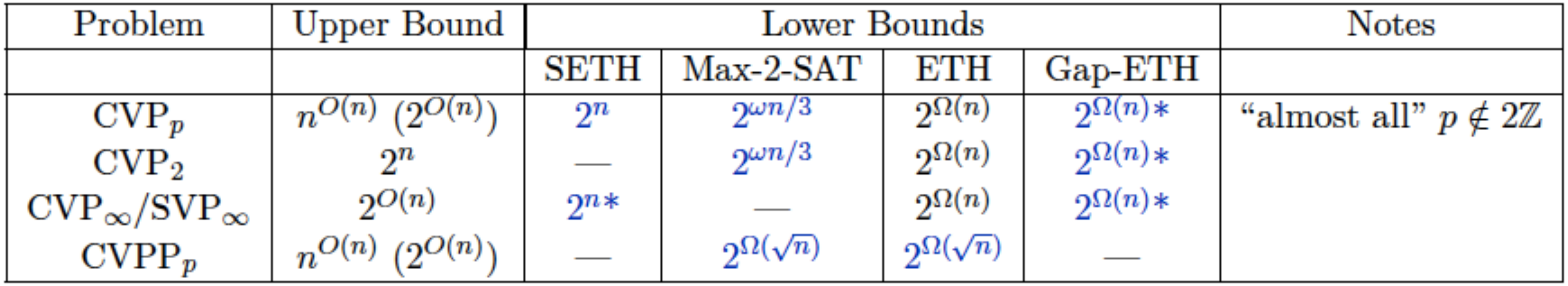

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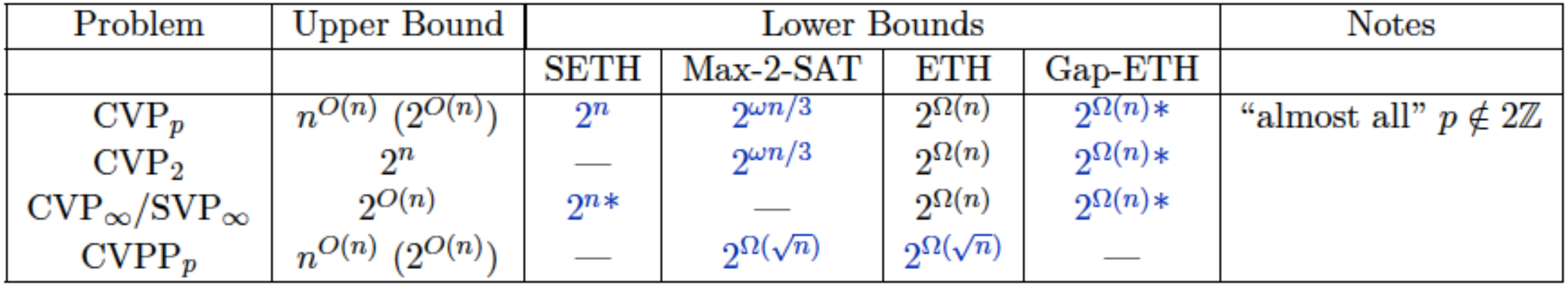

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- d >> n.

# Break?

# Act 3:
# What about SVP?

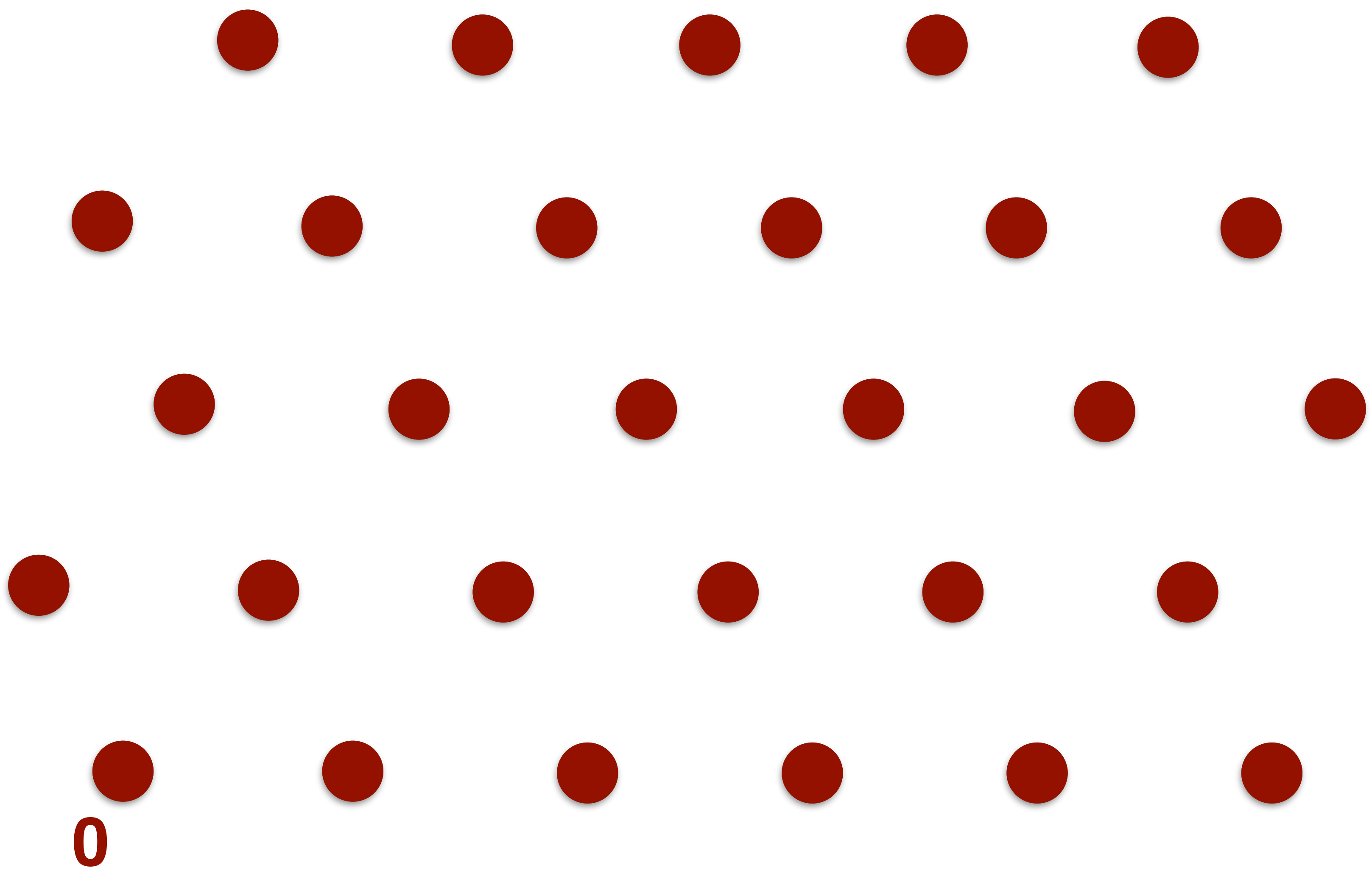
Divesh Aggarwal

Centre for Quantum Technologies

NUS National University of Singapore

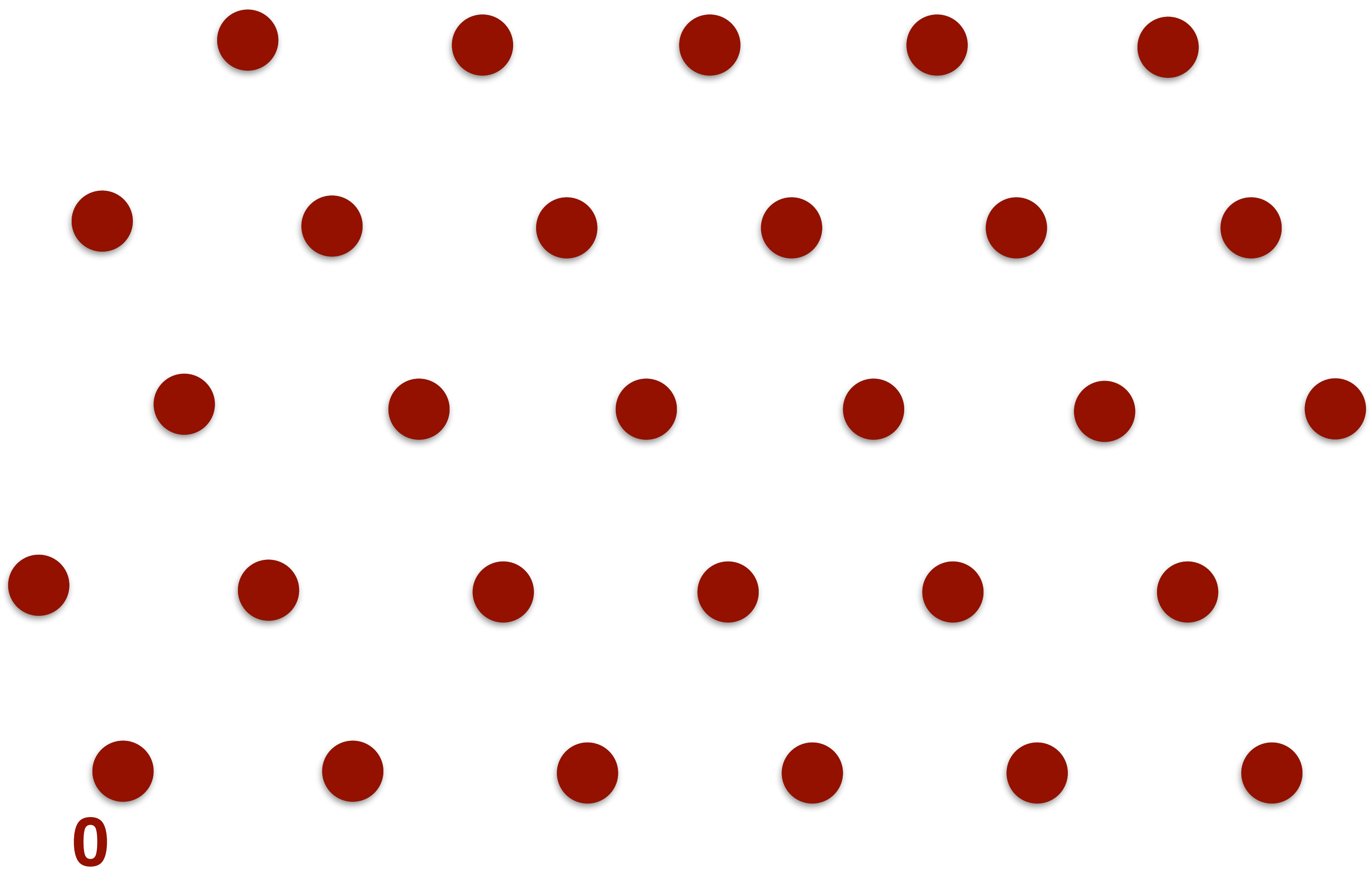
# The Shortest Vector Problem

# The Shortest Vector Problem



# The Shortest Vector Problem

$$\lambda_1^{(p)}(\mathcal{L}) := \min_{\mathbf{y} \in \mathcal{L}_{\neq \mathbf{0}}} \|\mathbf{y}\|_p$$



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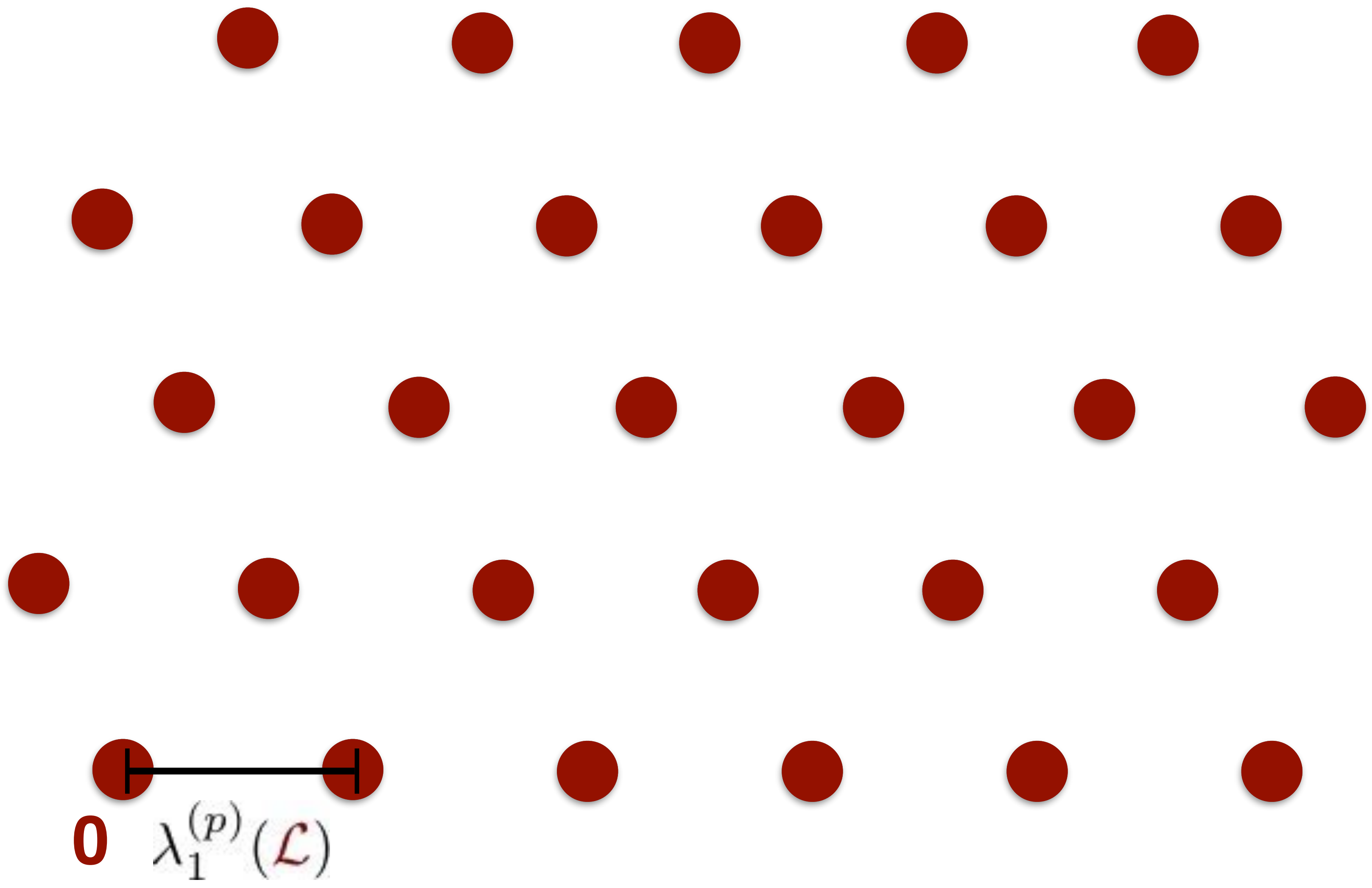


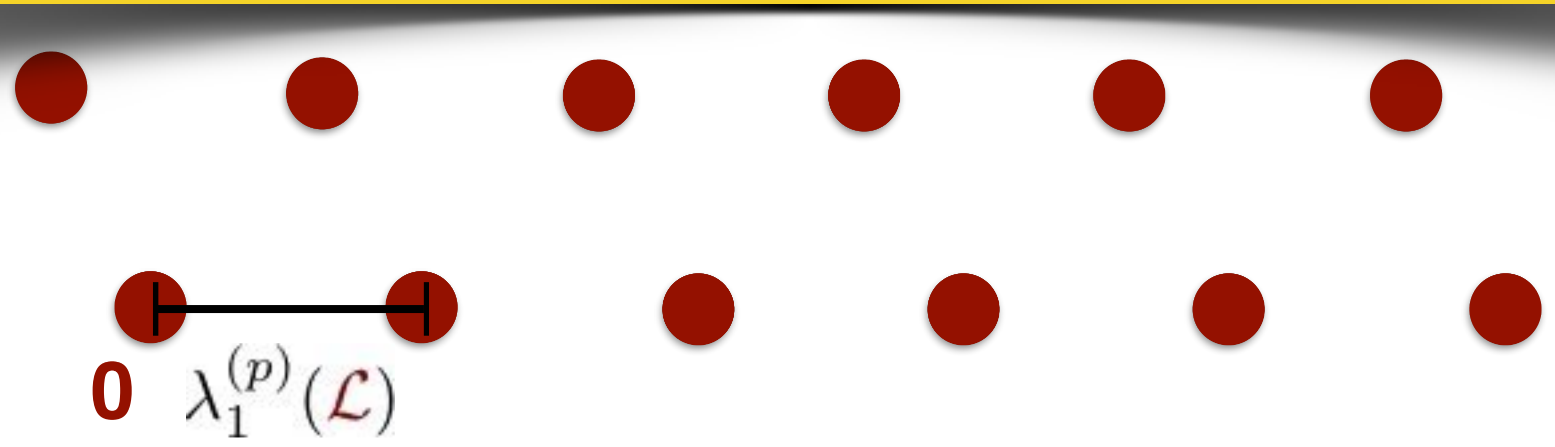
# The Shortest Vector Problem

$$\lambda_1^{(p)}(\mathcal{L}) := \min_{\mathbf{y} \in \mathcal{L}_{\neq \mathbf{0}}} \|\mathbf{y}\|_p$$

SVP$_p$ is the computational problem that asks us to compute $\lambda_1^{(p)}(\mathcal{L})$.

Approximate SVP$_p$ asks us to approximate $\lambda_1^{(p)}(\mathcal{L})$.

(We'll switch freely between the search and decision problems.)



# SVP Algorithms
# (it's complicated…)

| p | | |
|---|---|---|
| All | $2^{O(n)}$ | [AKS01, BN09, AJ08, DPV11] |
| 2 | $2^{n+o(n)}$ | [ADRS15, AS18] |
| 2 | $2^{n/2+o(n)}$ | 2-approx [ADRS15] |
| 2 | $n^{O(n)}$ (but fast) | (n=150!) [KT17] |
| 2 | $(3/2)^{n/2+o(n)} \approx 2^{0.29n}$ | Heuristic [BDGL15] |
| ∞ | $\approx 3^d$ | [DM18] |
| ∞ | $2^{0.62d}$ | Heuristic [DM18] |

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- Answered in 1998 by Ajtai. (Yes.)

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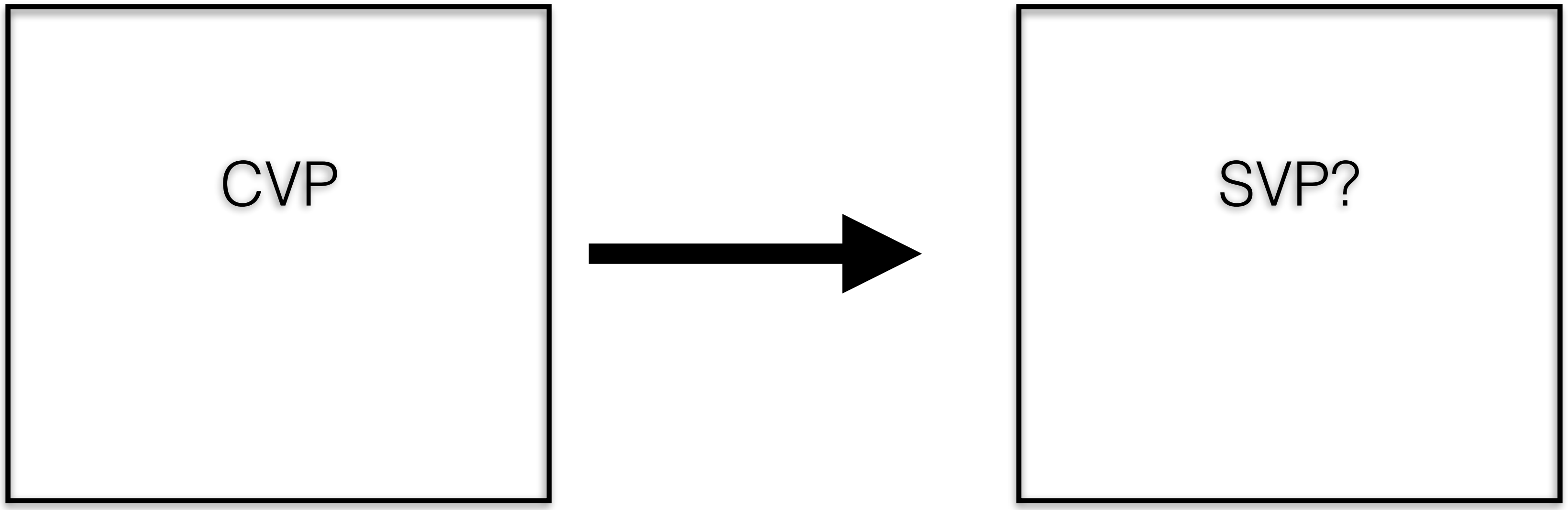
    - $\gamma = n^{c/\log\log n}$
- All known reductions are randomized [Mic12]

# Hardness of SVP: Dream Proof



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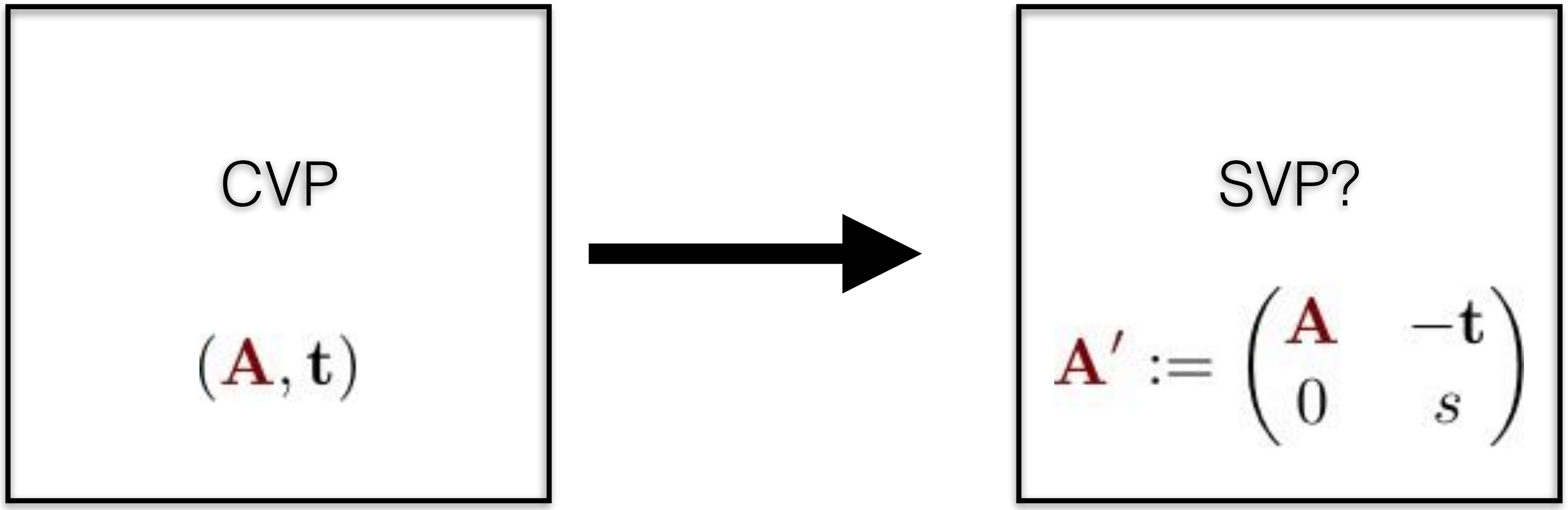

CVP

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SVP?

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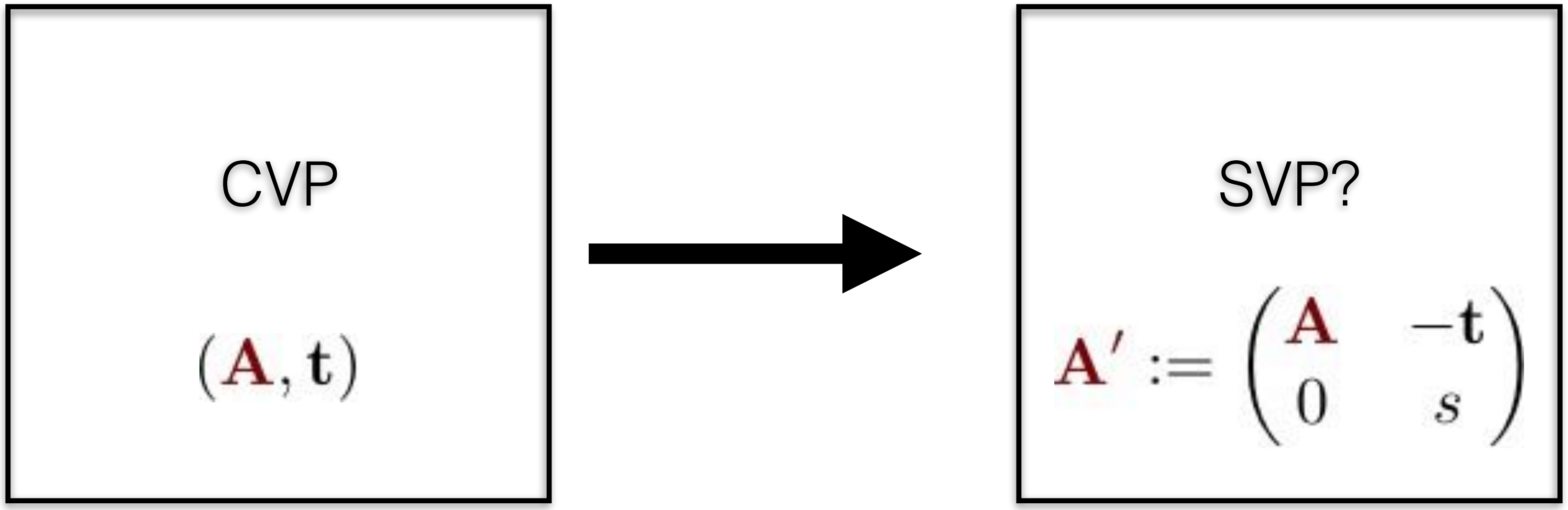

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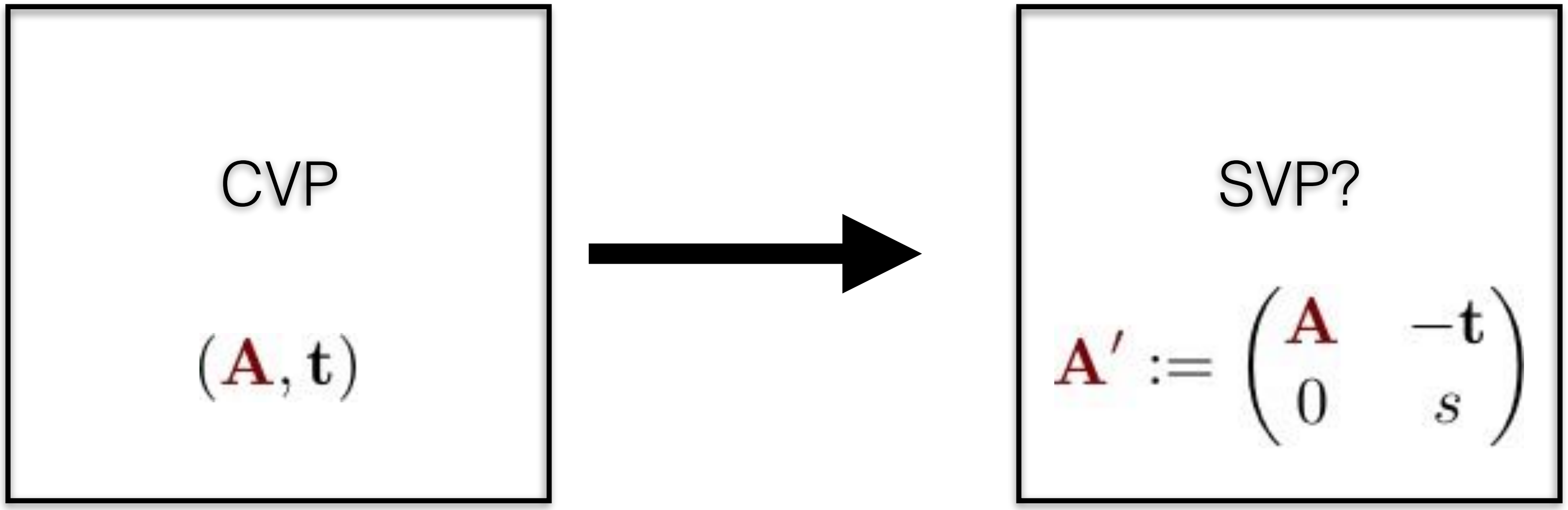

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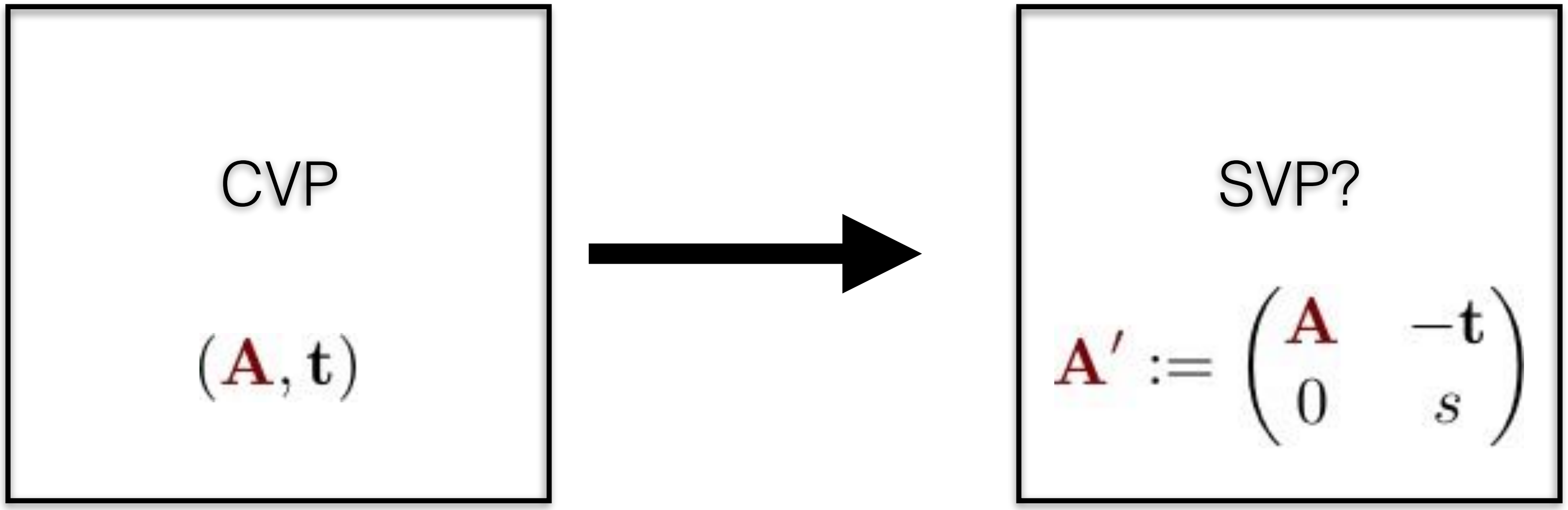

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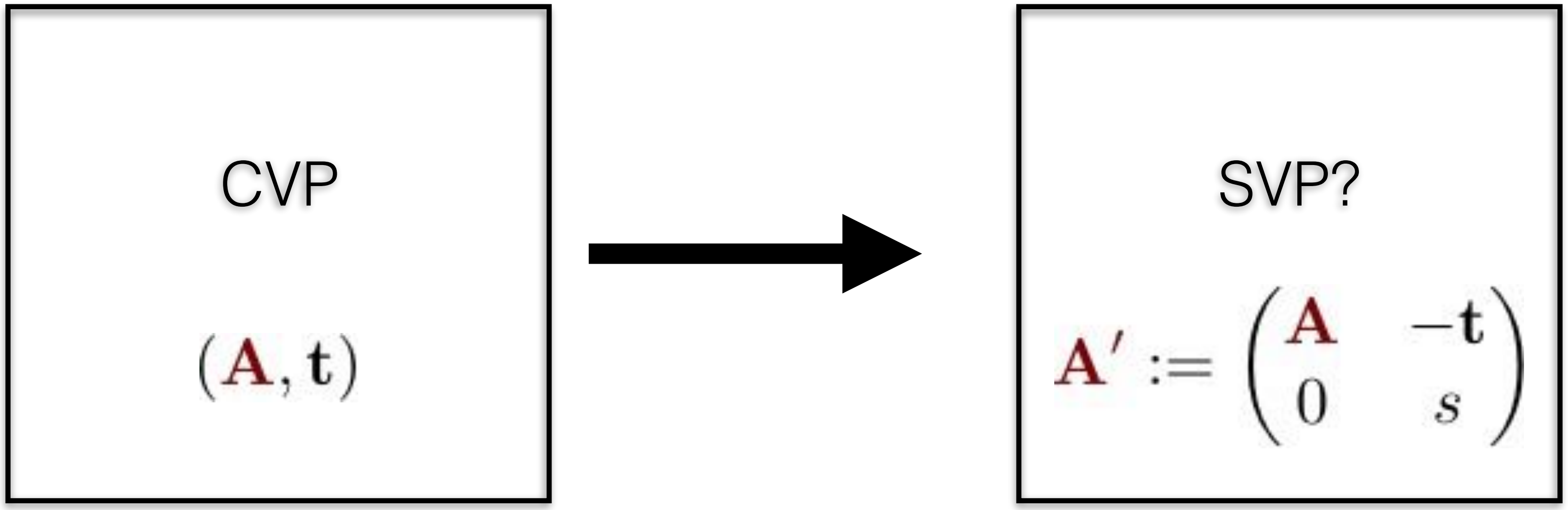

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Really only $k = 0$ is a problem. I.e., short vectors in $\mathcal{L}(\mathbf{A})$.

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It suffices to show hardness of CVP with more close vectors than short vectors.

(Note: The resulting lattice looks a lot like the lattices used in cryptography.)

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In order to get "many more close vectors than short vectors", we want

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(Our hard CVP instance from before is basically just
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All hardness reductions for SVP use some gadget like this. We show that any such gadget implies hardness.

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To prove $2^{\Omega(n)}$-hardness, we need $n^\dagger = O(n)$. I.e.,

$$. \quad \frac{N_p(\mathcal{L}(\mathbf{A}^\dagger), r^\dagger; \mathbf{t}^\dagger)}{N_p(\mathcal{L}(\mathbf{A}^\dagger), r^\dagger)} \geq 2^{\Omega(n^\dagger)} .$$

# Building the Gadget
# p > 2.14

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We just need to study the number of integer vectors in $\ell_p$ balls.

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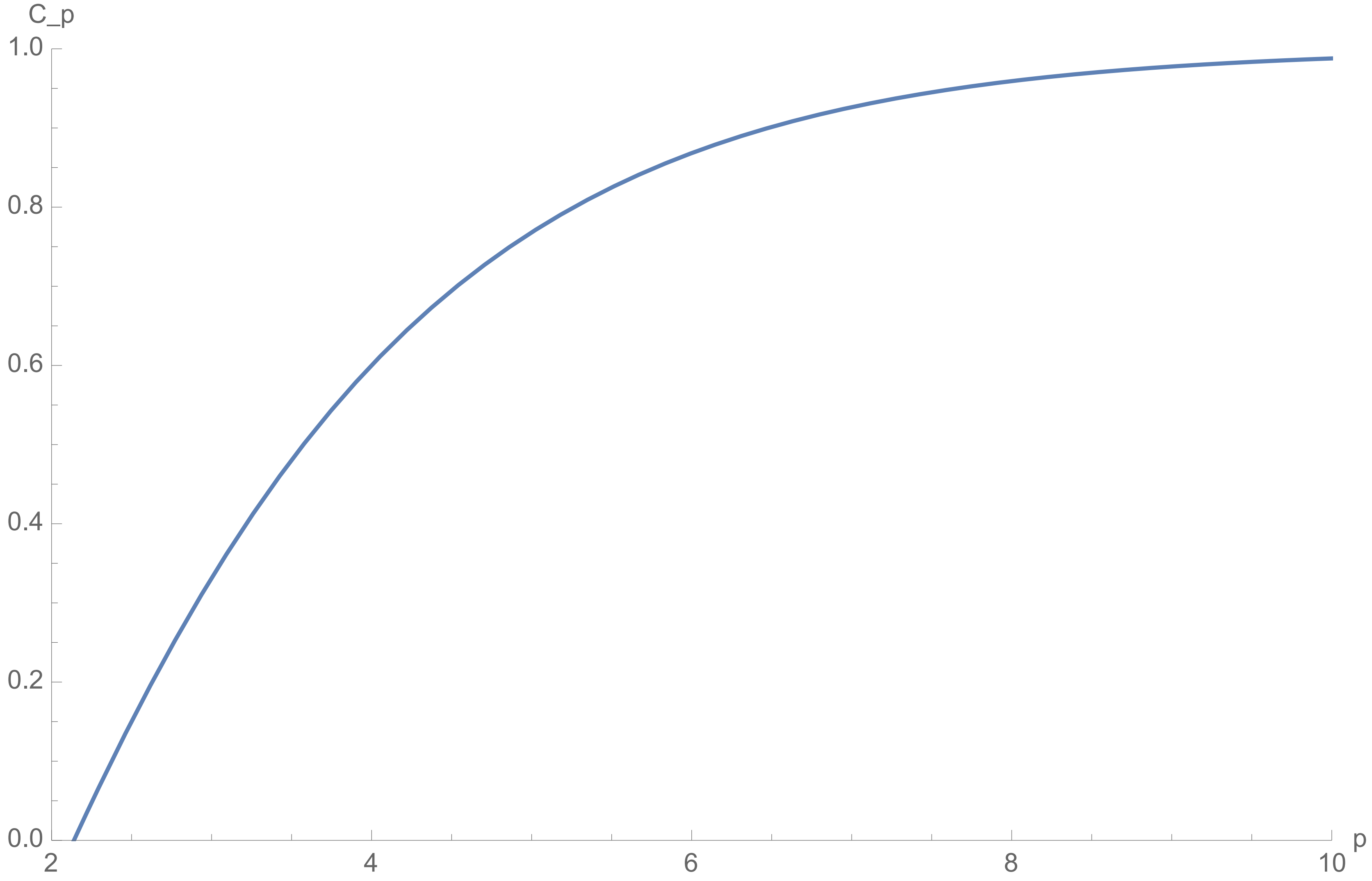
Technique due to [Mazo, Odlyzko 90] and [EOR91].

# SETH-Hardness for p > 2.14 via the Integer Lattice

No $2^{C_p n}$-time algorithm for $\mathsf{SVP}_p$ unless SETH fails.

(For "almost all" $p \gtrsim 2.14$.)



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$2^{\Omega(n)}$-hardness assuming no $2^{o(n)}$-time algorithm for Gap-2-SAT.

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(To analyze this, we study $\Theta_p(\tau; t^\dagger) := \sum_{z \in \mathbb{Z}} e^{-\tau|z - t^\dagger|^p}$.)

$2^{\Omega(n)}$-hardness assuming no $2^{o(n)}$-time algorithm for Gap-2-SAT.

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(The integer lattice can't work for $p \leq 2$.)

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SVP$_2$ is $2^{\Omega(n)}$-hard unless GapETH fails!

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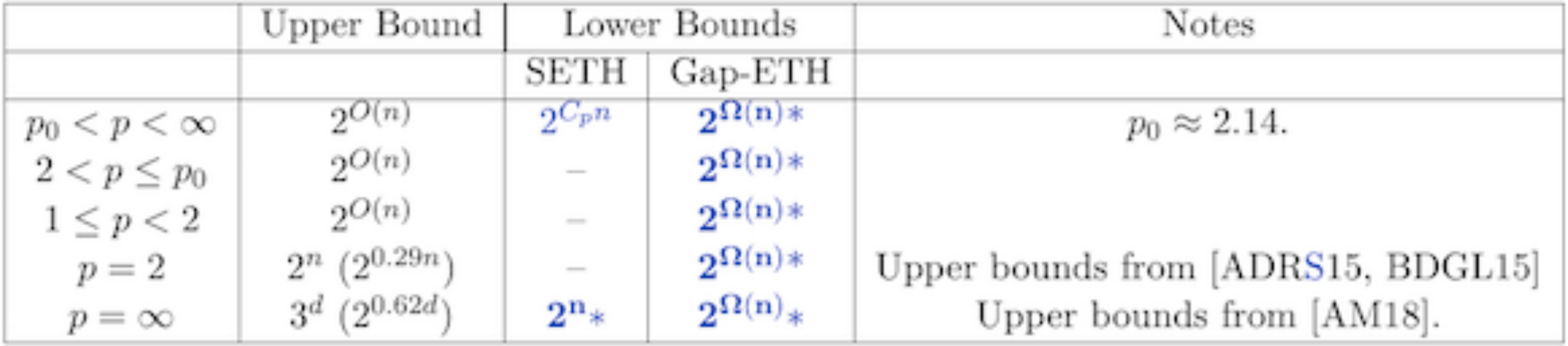
# Summary

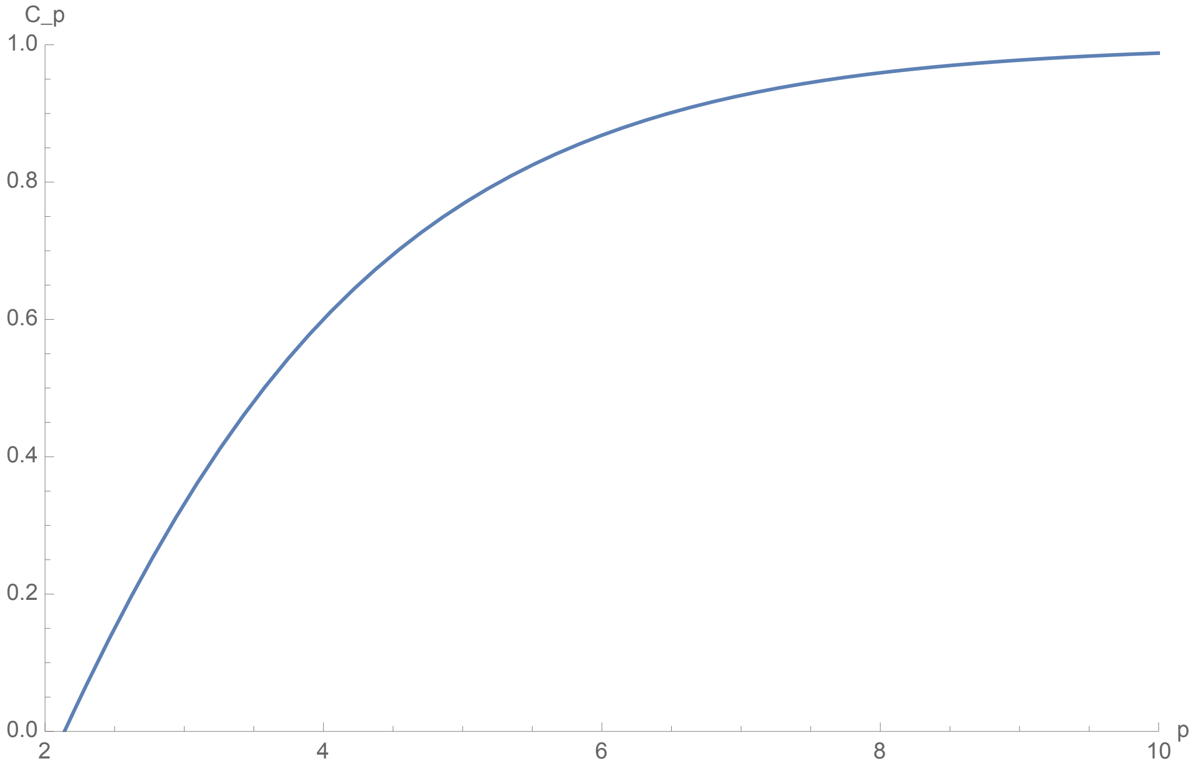


| | Upper Bound | Lower Bounds | | Notes |
|---|---|---|---|---|
| | | SETH | Gap-ETH | |
| $p_0 < p < \infty$ | $2^{O(n)}$ | $2^{C_p n}$ | $2^{\Omega(n)}*$ | $p_0 \approx 2.14$. |
| $2 < p \le p_0$ | $2^{O(n)}$ | – | $2^{\Omega(n)}*$ | |
| $1 \le p < 2$ | $2^{O(n)}$ | – | $2^{\Omega(n)}*$ | |
| $p = 2$ | $2^n$ $(2^{0.29n})$ | – | $2^{\Omega(n)}*$ | Upper bounds from [ADRS15, BDGL15] |
| $p = \infty$ | $3^d$ $(2^{0.62d})$ | $2^n*$ | $2^{\Omega(n)}*$ | Upper bounds from [AM18]. |

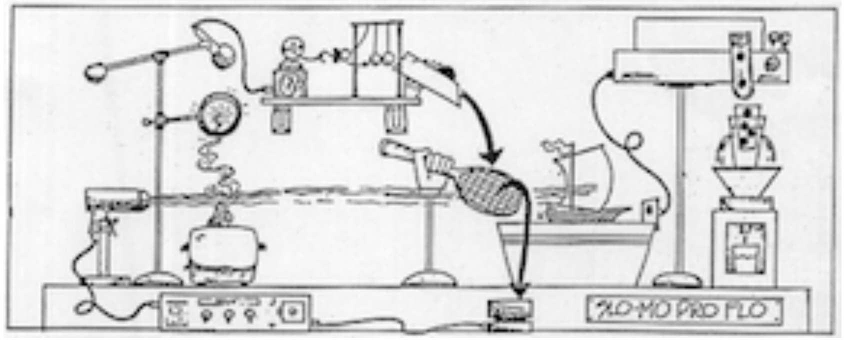
Blue = new result.

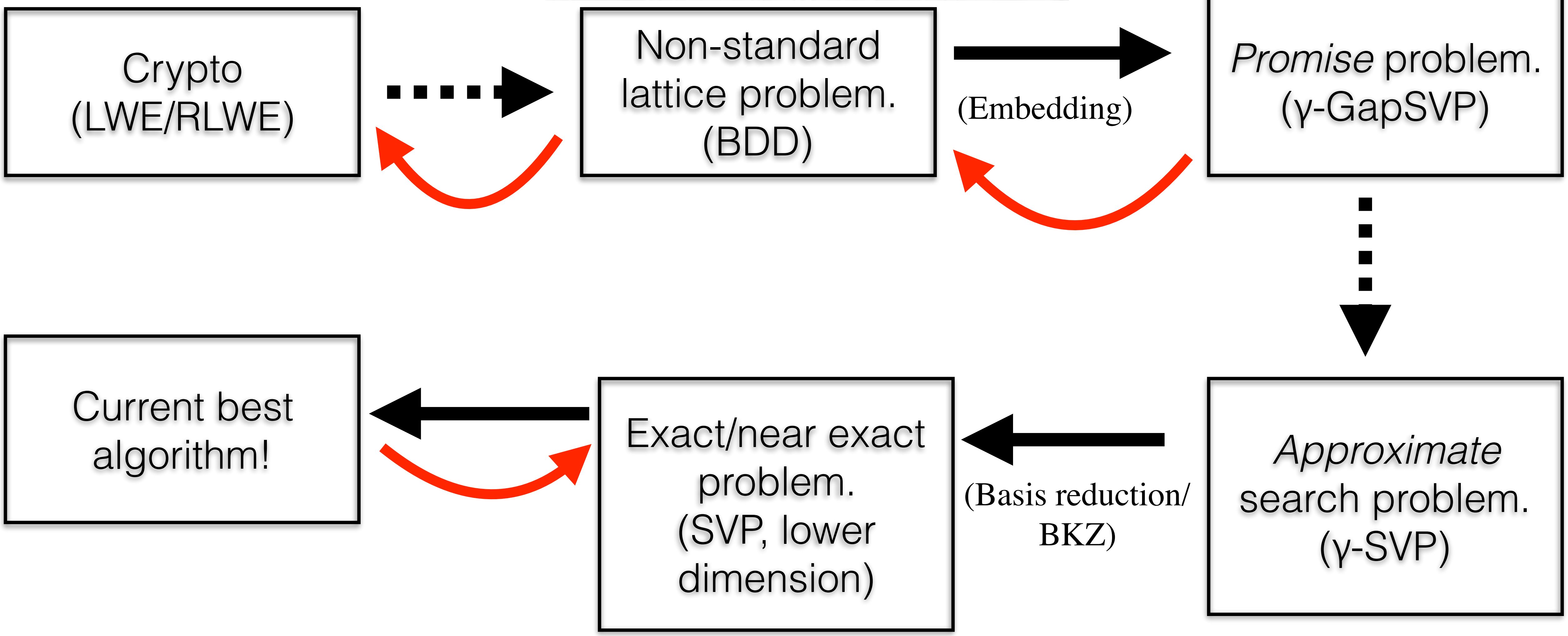
(…) = heuristic algorithm

* = hardness for some constant approximation factor



# The Path Forward





Crypto (LWE/RLWE)

Non-standard lattice problem. (BDD)

(Embedding)

*Promise* problem. (γ-GapSVP)

*Approximate* search problem. (γ-SVP)

(Basis reduction/ BKZ)

Exact/near exact problem. (SVP, lower dimension)

Current best algorithm!

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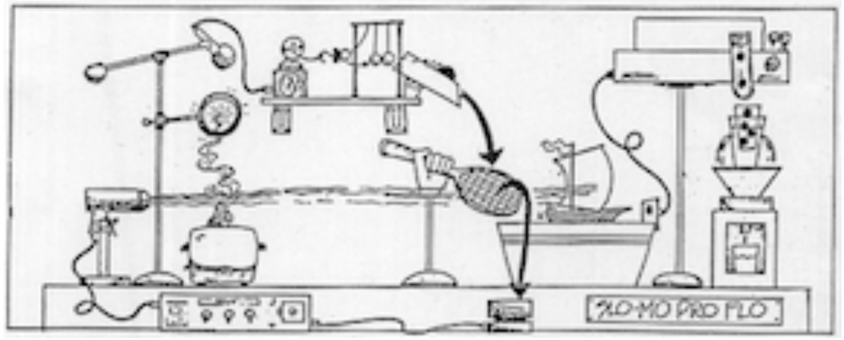

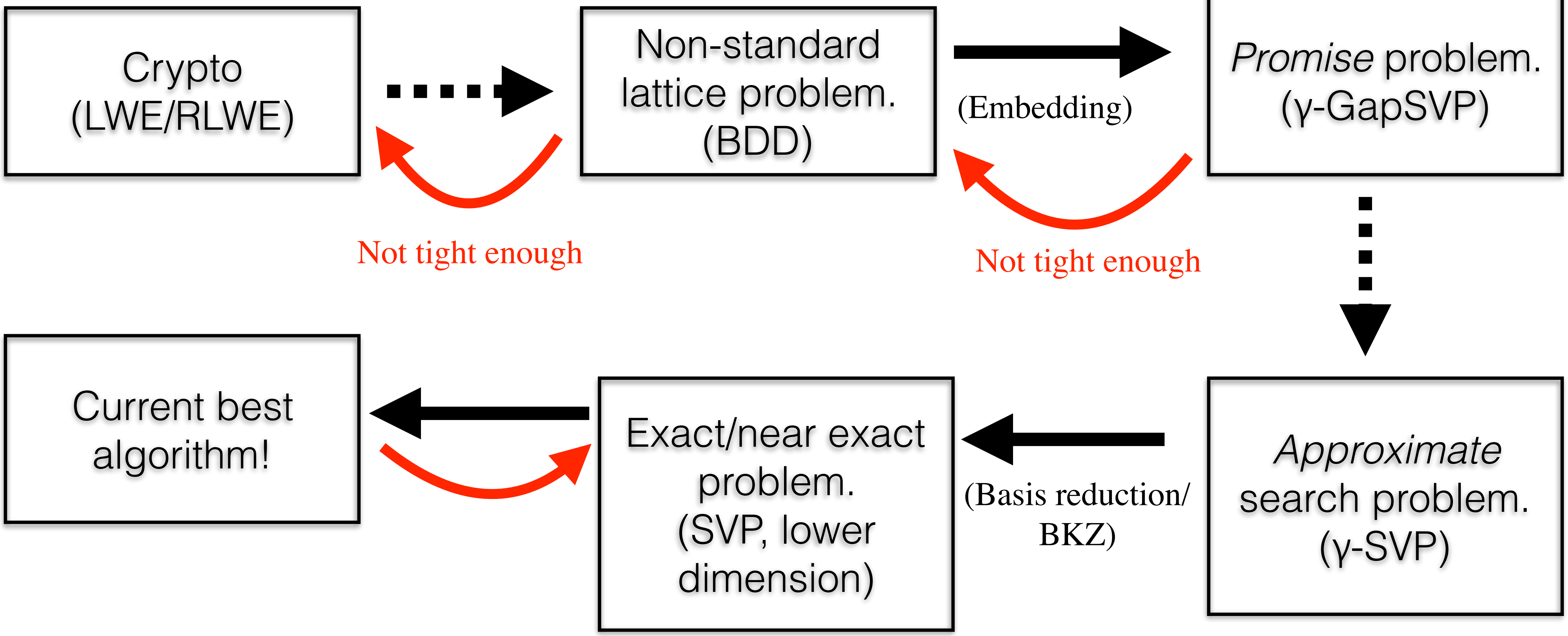

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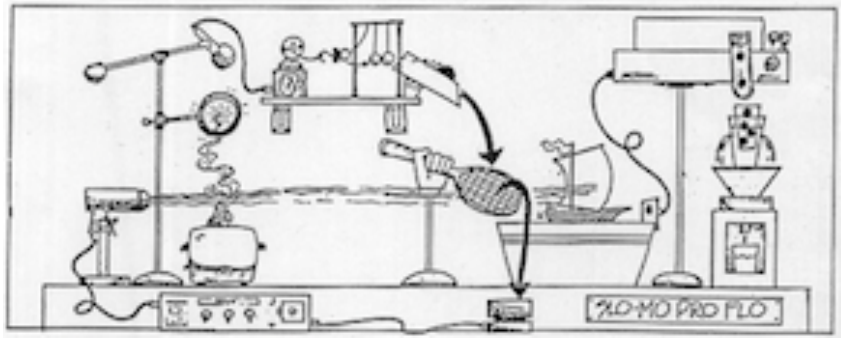

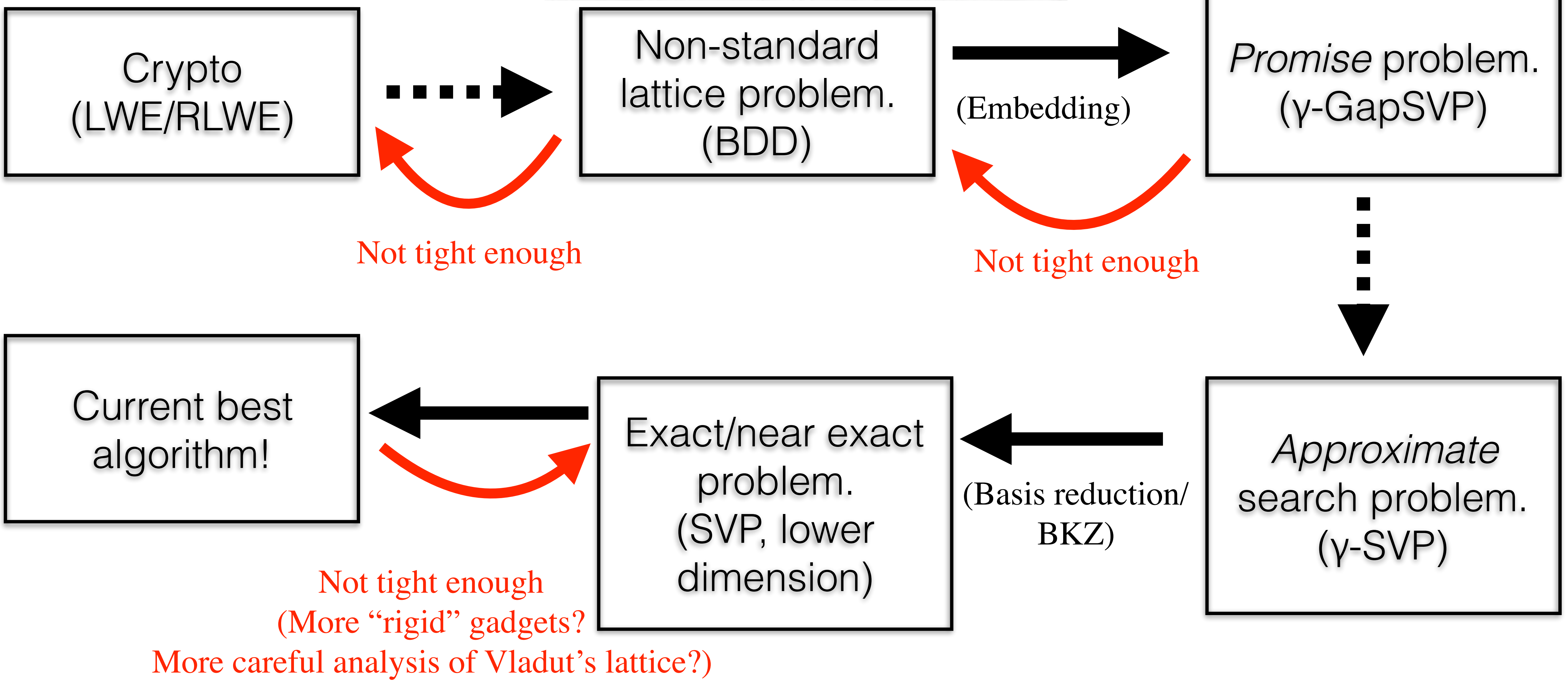

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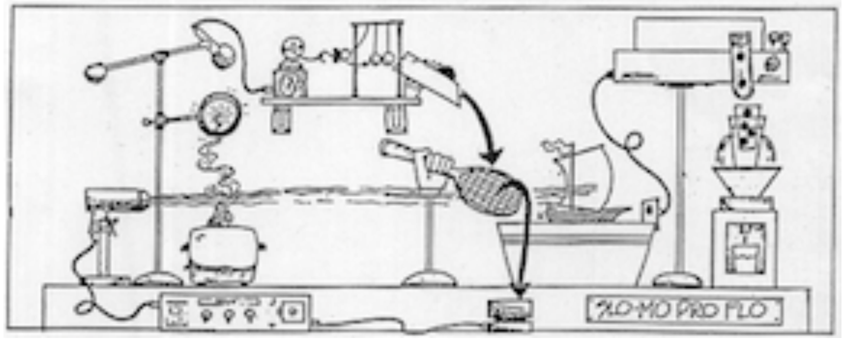

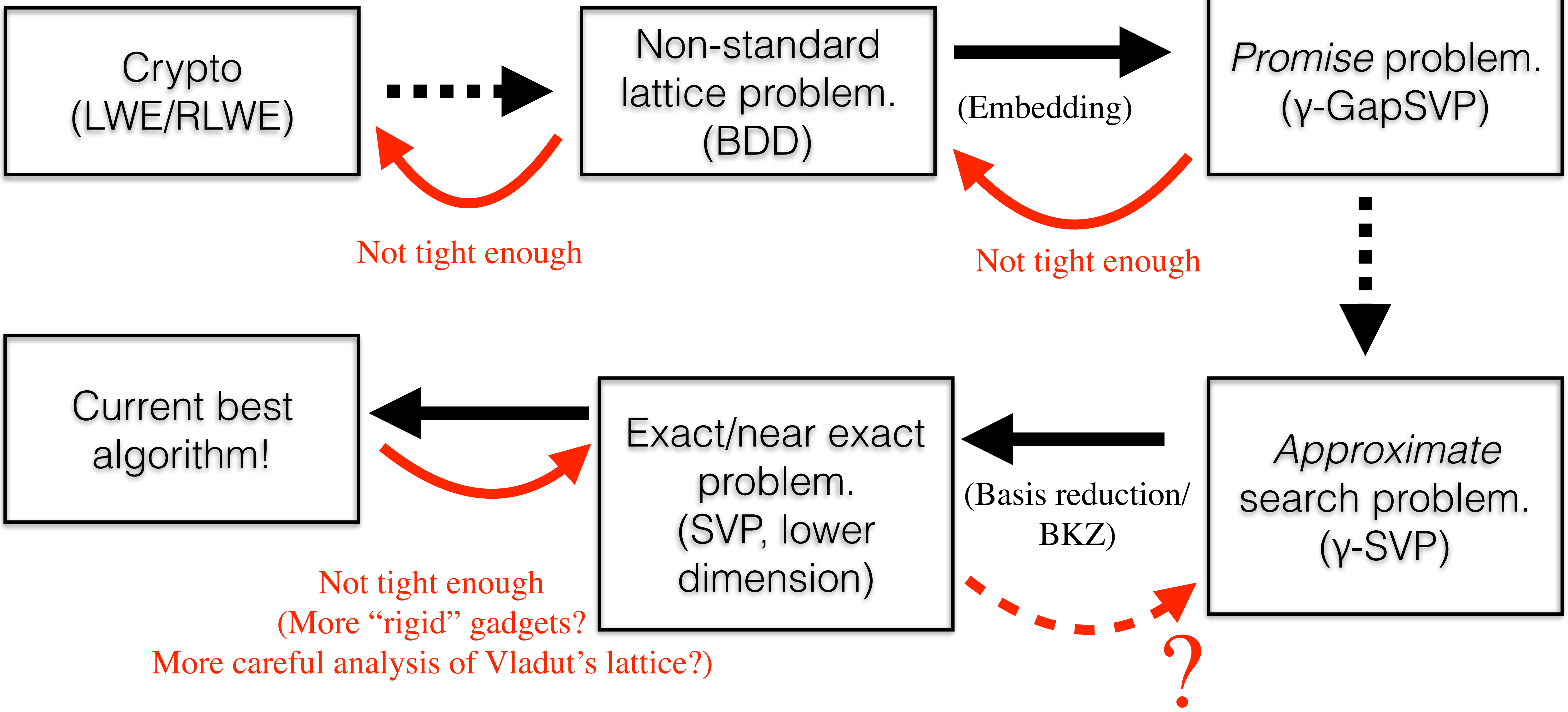

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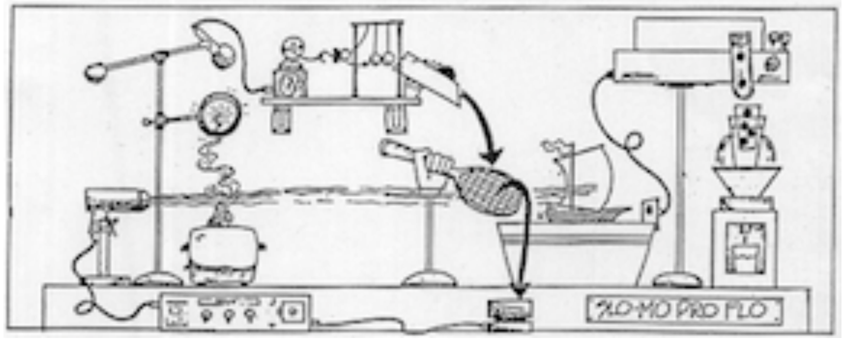

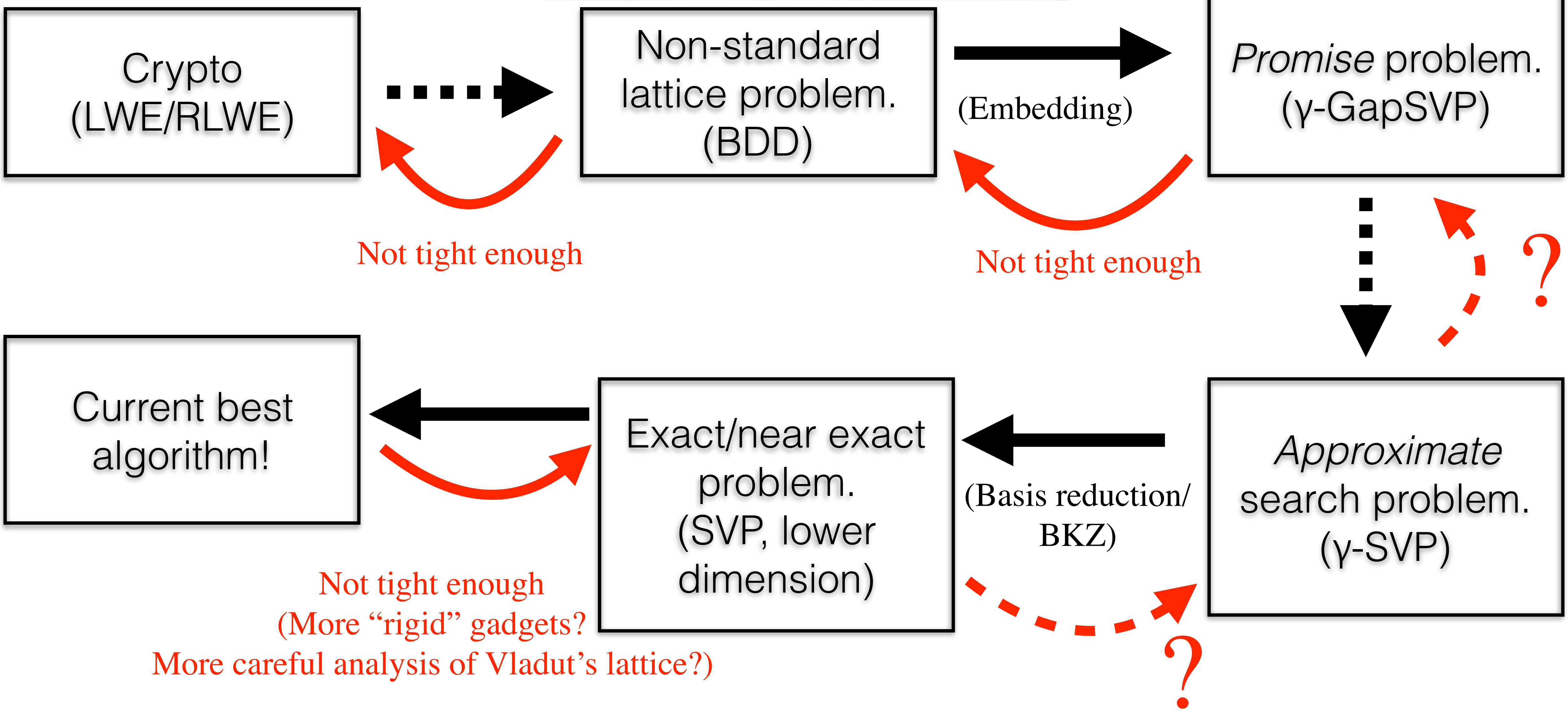

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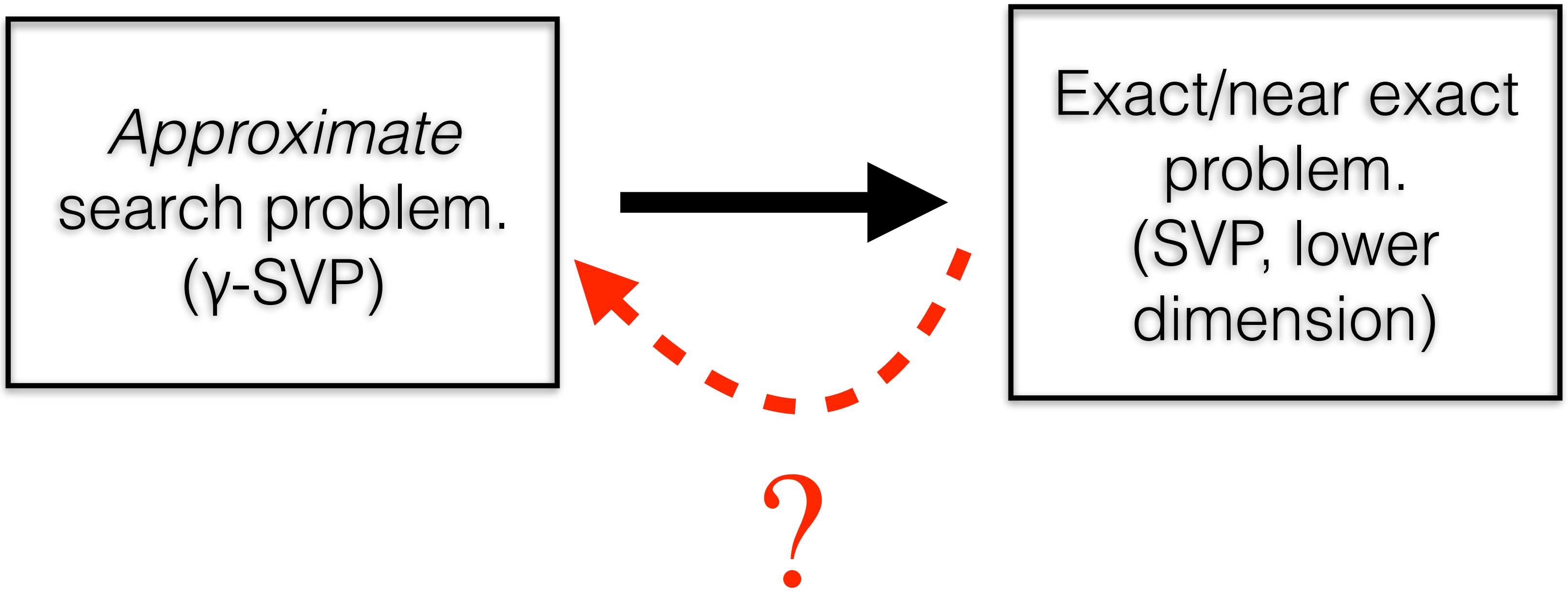
# Is BKZ Optimal?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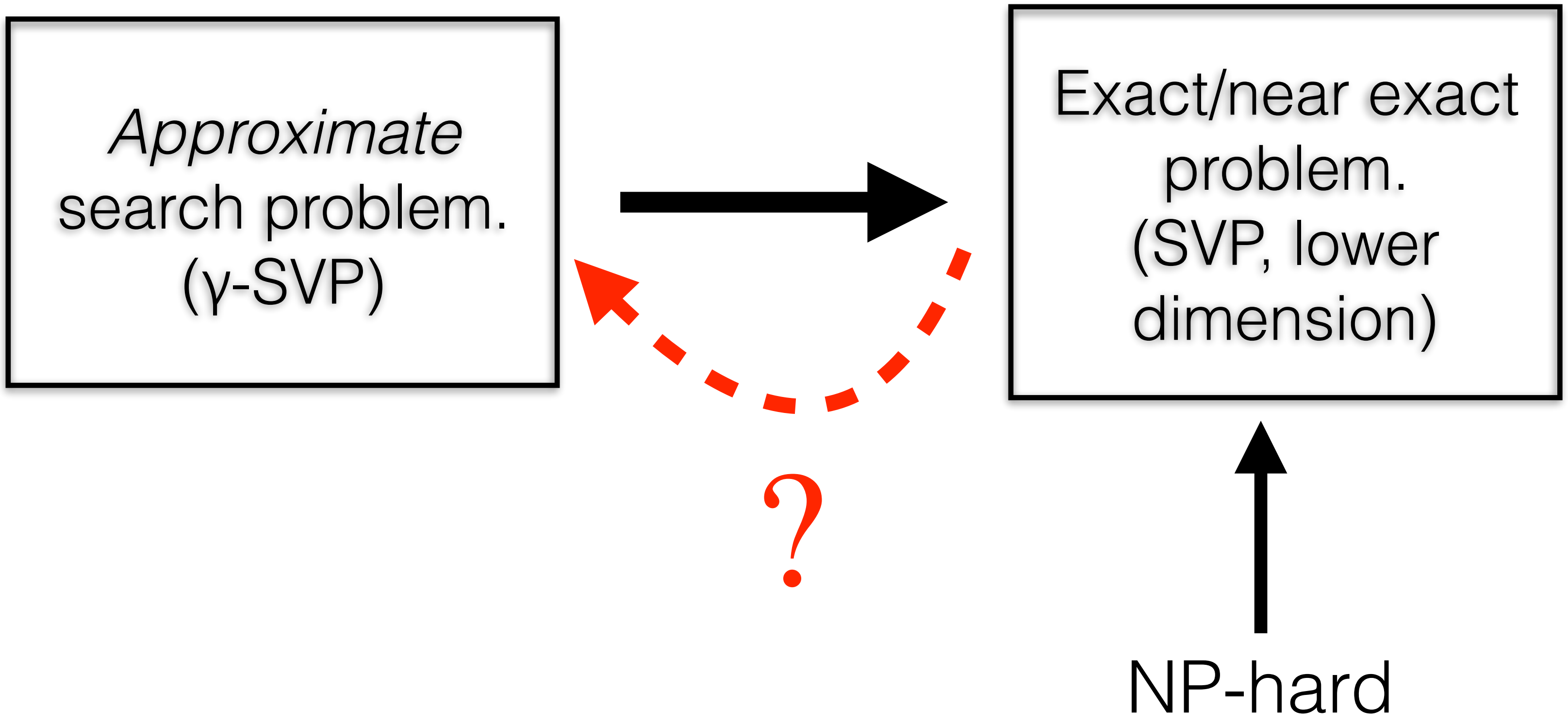

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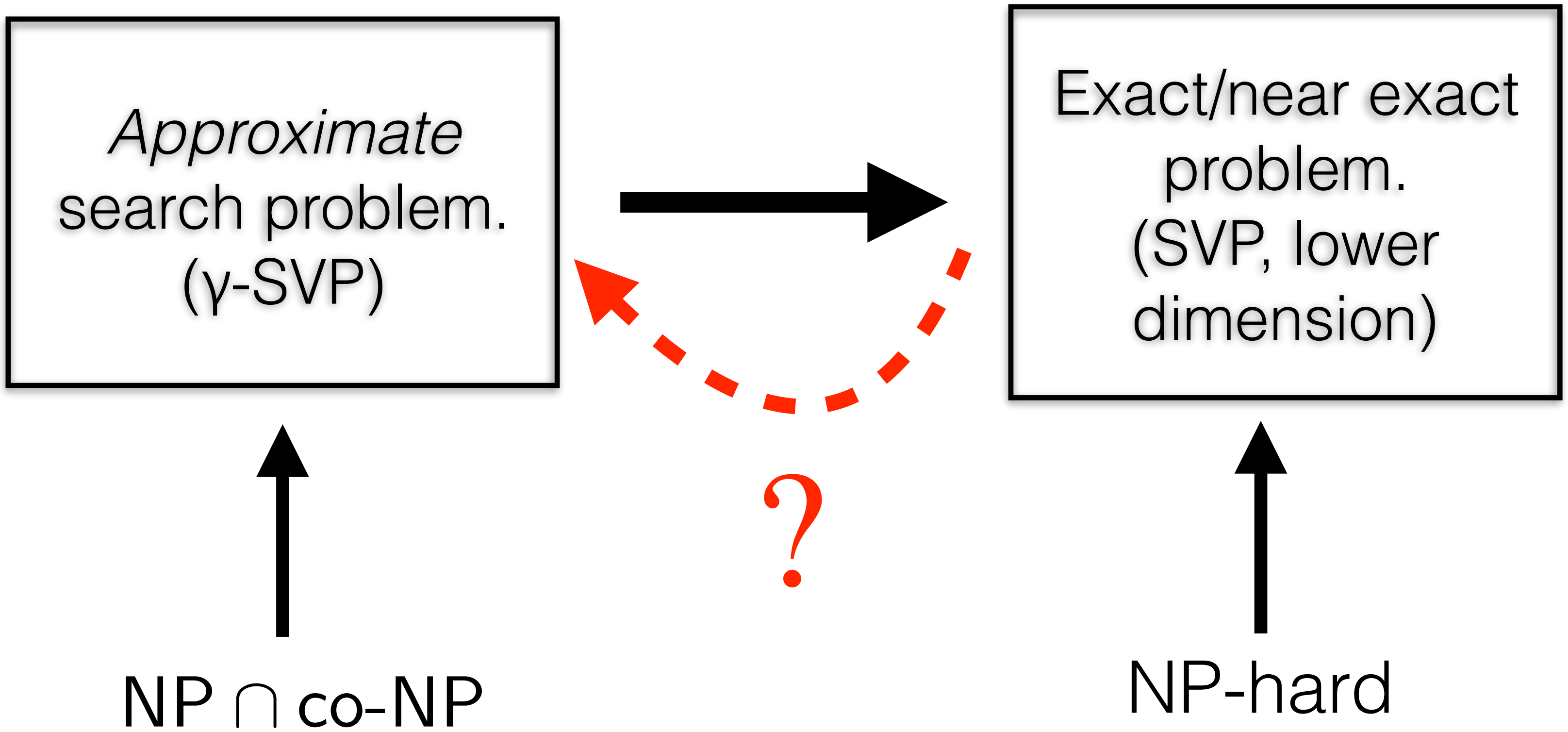

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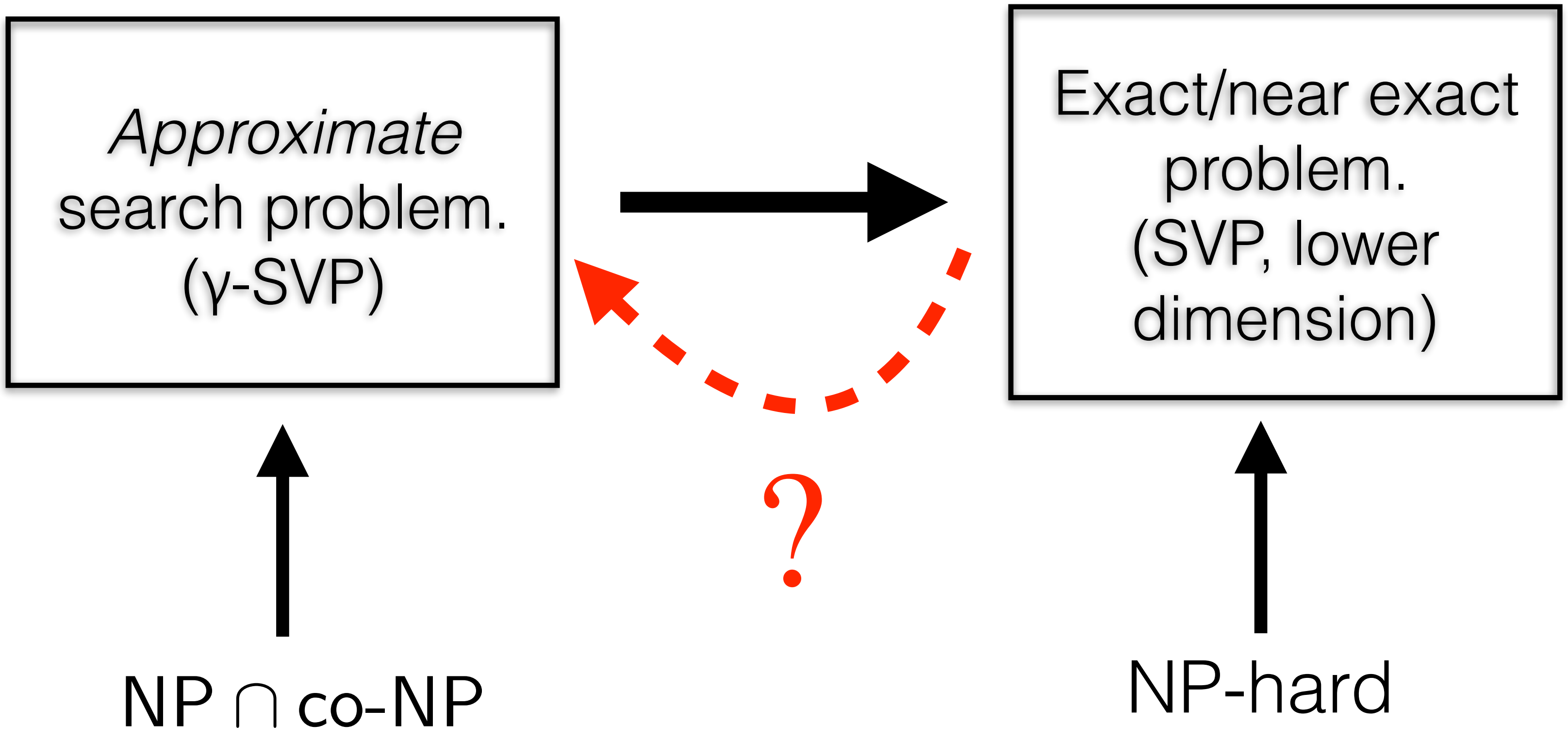
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Superpolynomial? Non-deterministic? Non-uniform?

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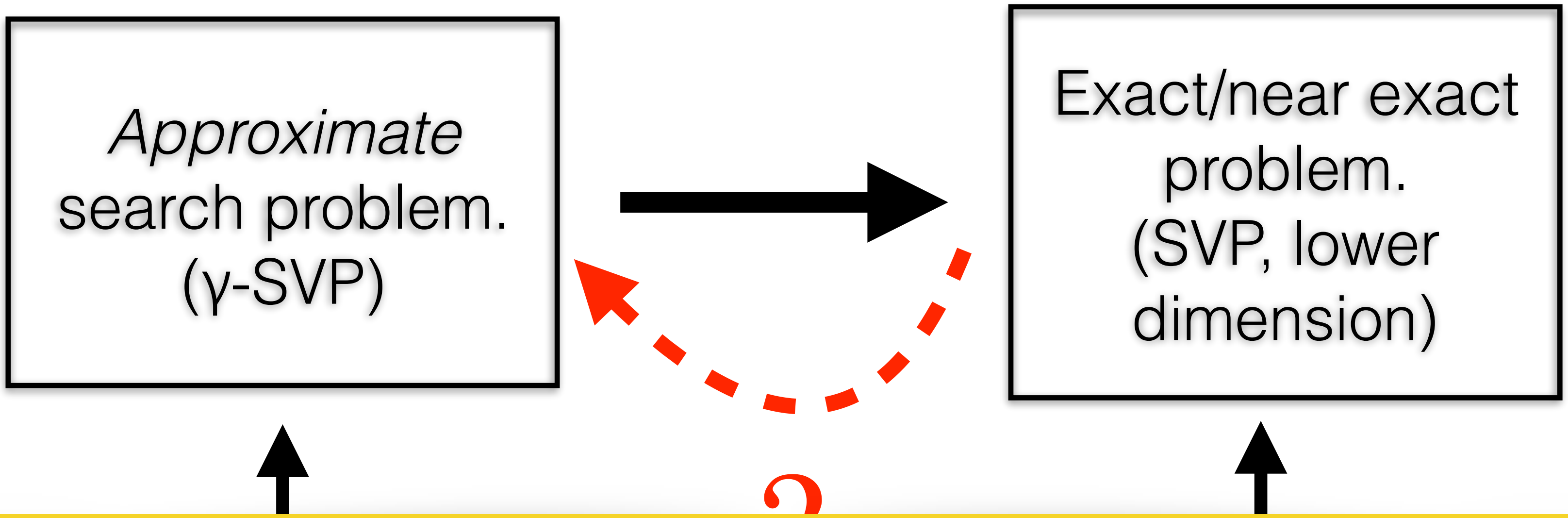

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Maybe BKZ is fundamentally the wrong approach for approximate lattice problems?

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# Thanks!