The Mersenne cryptosystem

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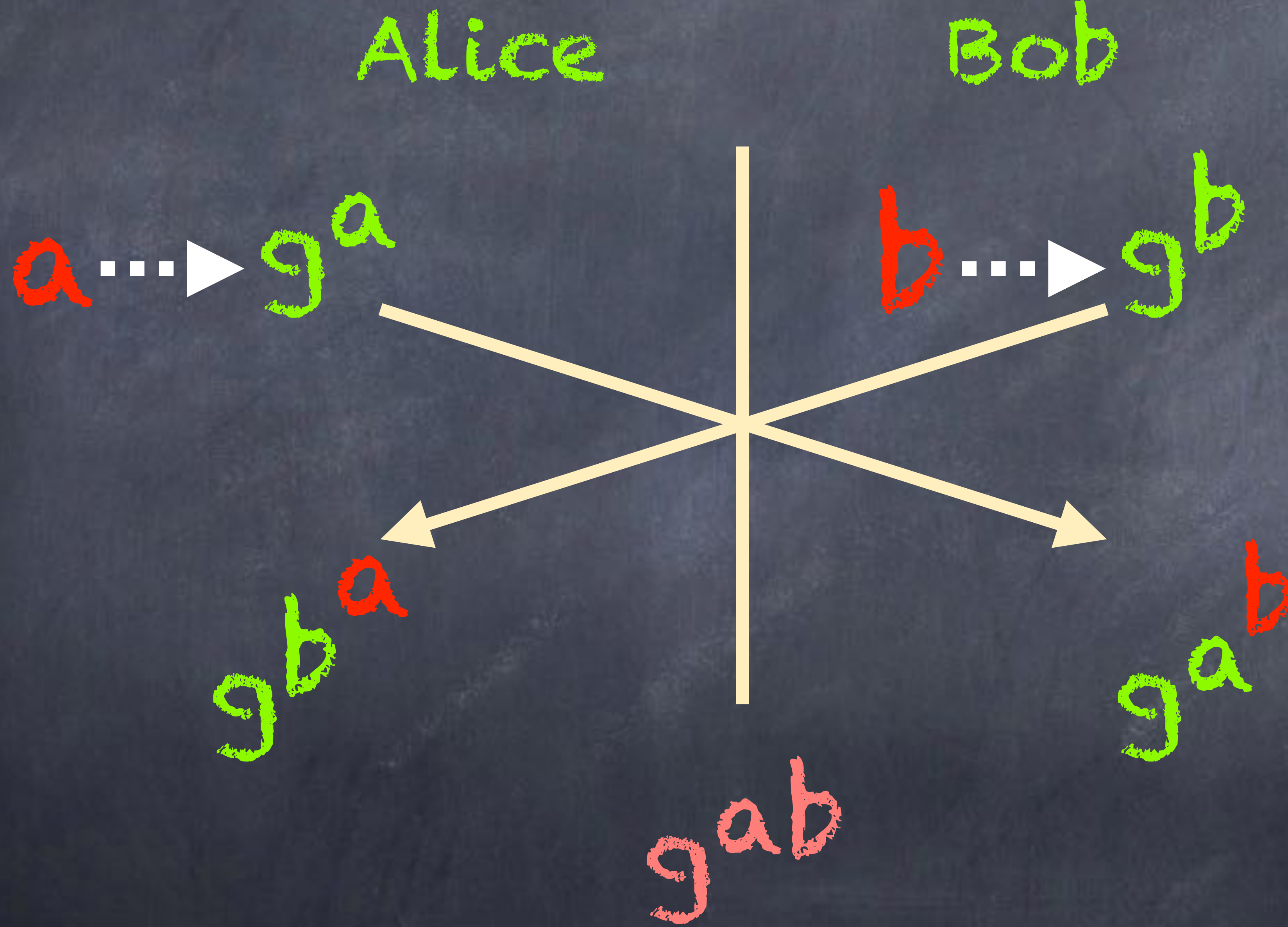
joint work with Divesh Aggarwal,
Anupam Prakash and Miklos Santha

Public key crypto
(Diffie-Hellman 1976)



g generator of a (large) cyclic group

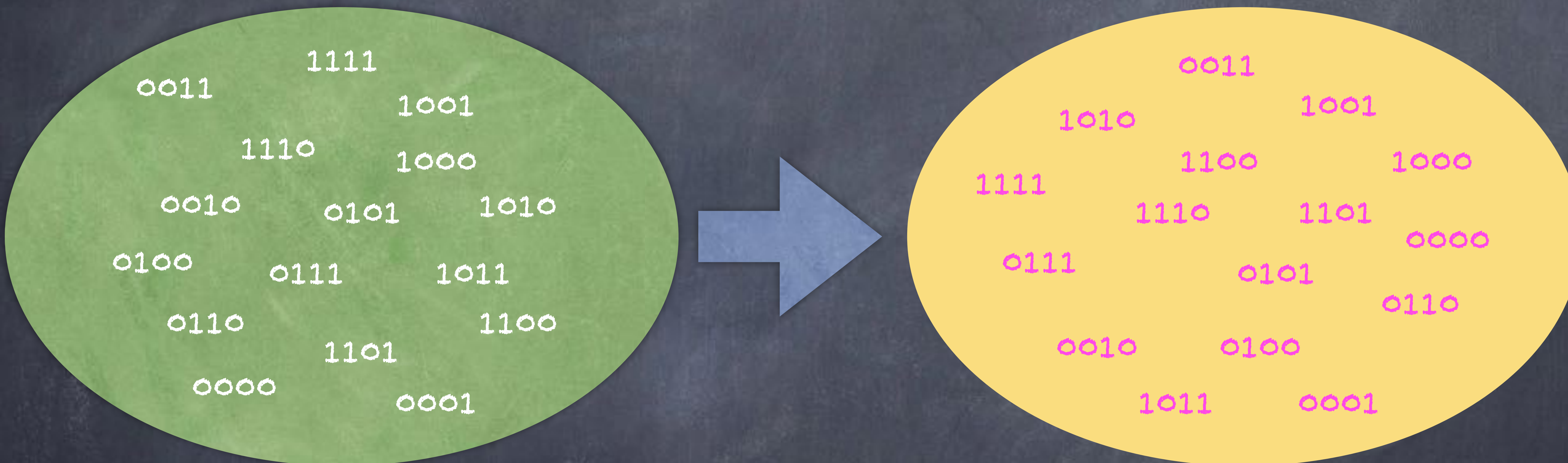
The threat of Quantum Computers

Quantum physics ?

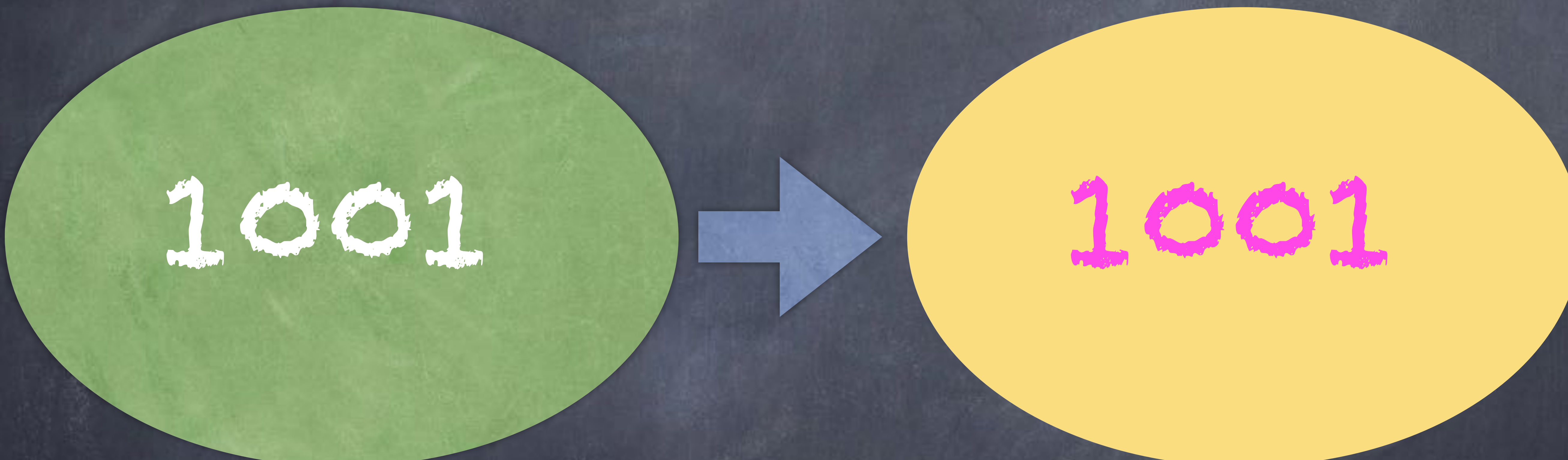
State superposition of a physical object

+

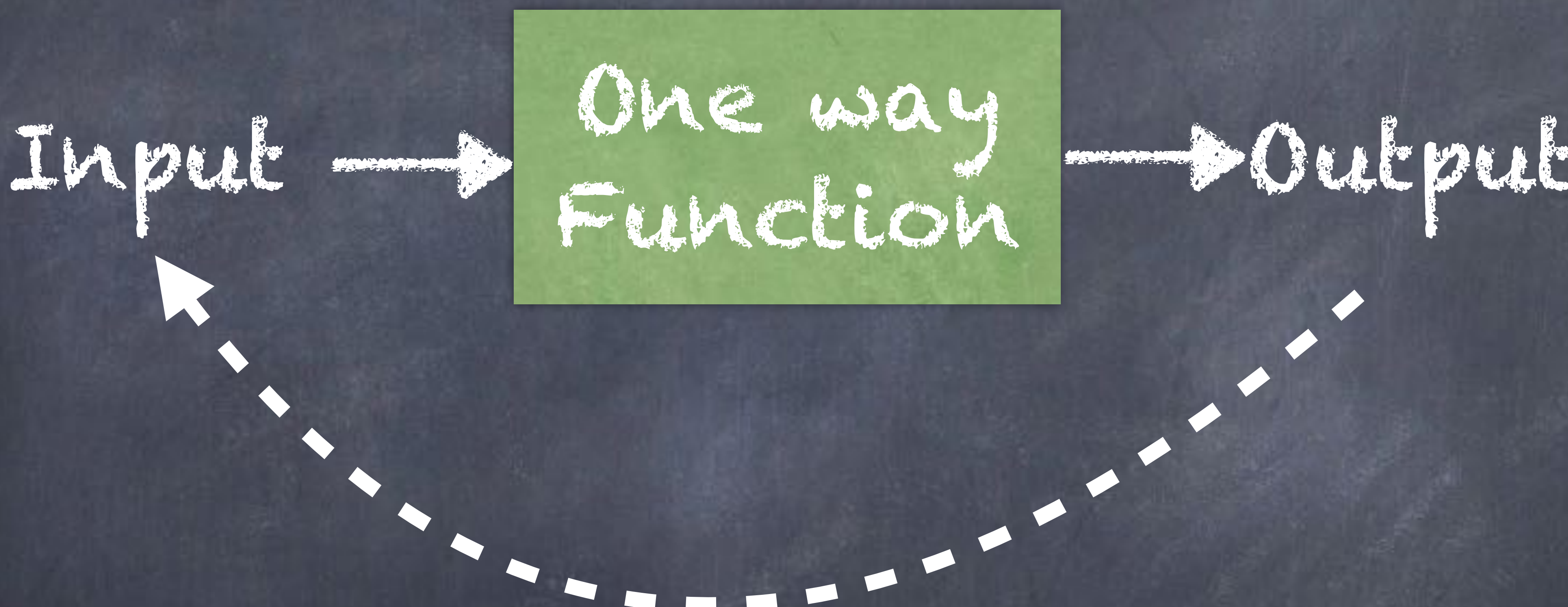
Quantum computer (compute phase)



Quantum computer (measurement phase)



Exhaustive Search



How to go back

For a « perfect » function : time N

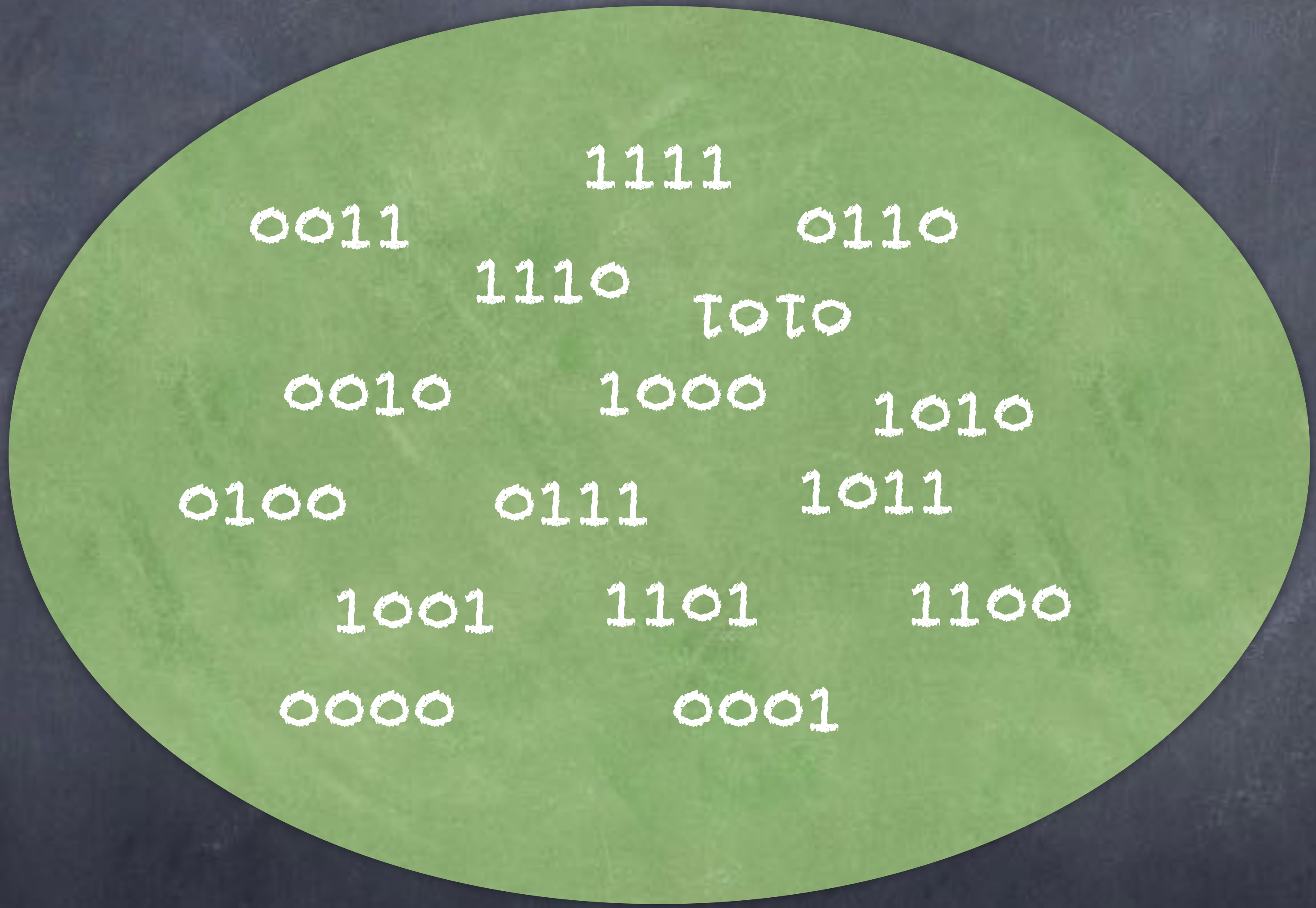
Post-quantum Era

A fast quantum mechanical algorithm for database search

Lov K. Grover
3C-404A, Bell Labs
600 Mountain Avenue
Murray Hill NJ 07974
lkgrover@bell-labs.com

Search within N elts in time Sqrt(N)
(even for a « perfect » function)

Grover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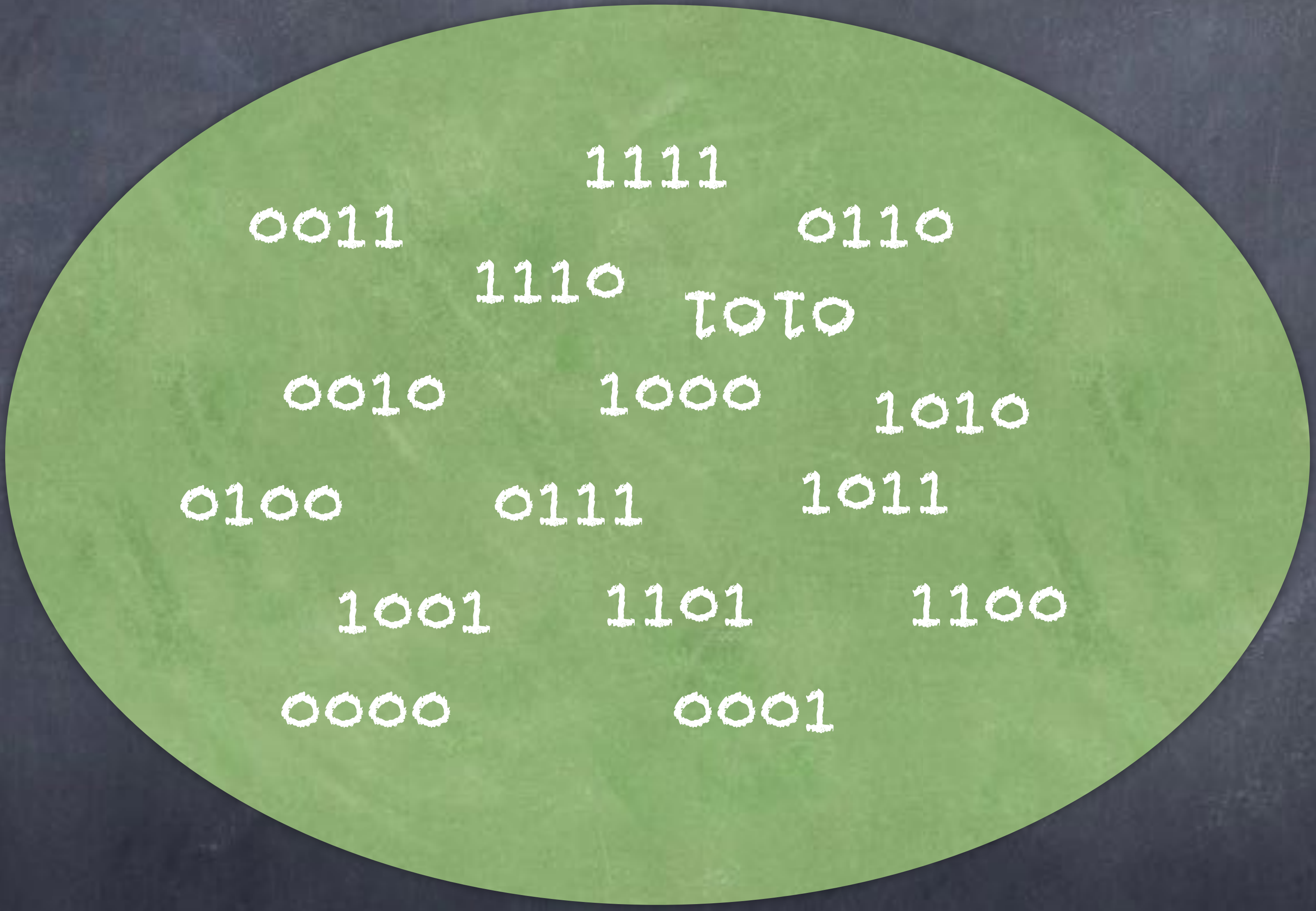
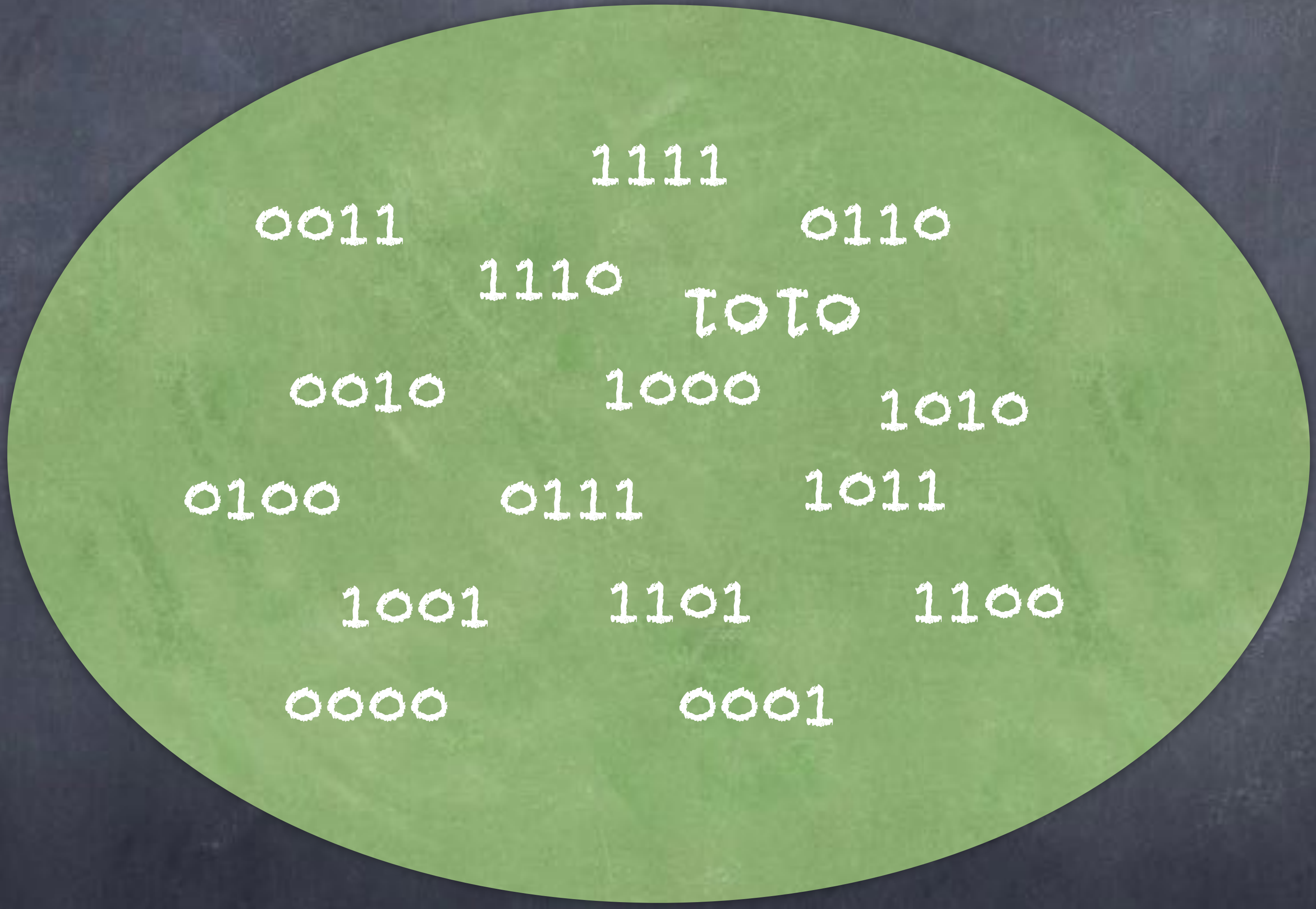
Grover



Running

Grover



Running

Grover

1111

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1110 TOTO

0010 1000 1010

0100 0111 1011

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Running .

Grover

Running ..

Grover

Running ...

Grover

ToTo

Consequences in crypto

Doubling the size of symmetric key !

Post-quantum Era

Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer*

Peter W. Shor†

arXiv:quant-ph/9508027v2 25 Jan 1996

Quantum Fourier transform

Consequences

- Diffie - Hellman
- RSA

No longer secure

Mersenne system

- Inside Ring and Noise family with
  - NTRU
  - Codes
  - Ideal Lattices, RLWE

- With a different Ring
  - Z/pZ (p Mersenne prime)

Mersenne ring and distance

- Ring Z/pZ
  - p a Mersenne prime, i.e., 2^n-1

Let :

- Rp(X)=rep of X in [0,p-1]

- HW(X)=num of 1 in binary of X

Some easy properties of arithmetic mod p

0) $X \equiv (X \bmod 2^n) + (X \operatorname{div} 2^n)$ [mod p]

1) $HW(X+Y) \leq HW(X) + HW(Y)$

2) $HW(XY) \leq HW(X) \times HW(Y)$

3) $HW(Rp(X)) \leq HW(X)$

4) $Rp(X) \neq 0 \Rightarrow HW(Rp(-X)) = n - HW(Rp(X))$

Warm Up

Single bit version

Mersenne basics (single bit version)

H=f/g [mod p]
(f and g containing few 1s, i.e. ≤h)

Encryption

a et b with few 1s

C=±(a H + b)

Decryption

gC=±[a f + b g]
nb 1 => ±

Mersenne basics (single bit version)

$p = 2^{31} - 1 = 2147483647 = 0x7FFFFFFF$
$H = f/g = 0x8002000/0x20000008$
$= 0x42E8BE0F$

Encryption

a=0x80800
b=0x40000080
C=±(a H + b)
=0x766CAB3A

Decryption

gC=0x110084A6
nb 1 = 8 (< 15) => +

Mersenne basics (single bit version)

Analysis of decryption

$$g(aH+b) \equiv af+bg \; [\text{mod } p]$$

$$\begin{aligned} HW(Rp(af+bg)) &\leq HW(a)HW(f)+HW(b)HW(g) \\ &\leq 2\,h^2 \leq n/2 \end{aligned}$$

$$\begin{aligned} HW(Rp(-(af+bg))) &= n - HW(Rp(af+bg)) \\ &\geq n/2 \end{aligned}$$

Multi-bit Mersenne

underlying encryption

Mersenne basics
Change key for more bits

$H = f/g \iff f(-1/H) + g = 0$ [mod p]

I.e. $f R + g = 0$

$T = f R + g$ [mod p]
(R fully random)

Mersenne
(basic multi-bit encrypt)

$T = fR + g$ [mod p] (R fully random)

Encryption

$C1 = a\ R + b1$

$C2 = a\ T + b2$

$E(m) = (C1, C2 \oplus Enc(m))$

Decryption of $(C1, Z)$

$C2' = f\ C1$

$m = Dec(C2' \oplus Z)$

Enc and Dec : Encoding / Decoding

Mersenne
(basic multi-bit encrypt)

Analysis of decryption

$C2 = afR + (ag+b2)$
$C2' = afR + b1\,f$

$\text{Hdist}(C2, C2') \leq \text{Hdist}(C2, afR) + \text{Hdist}(C2', afR)$

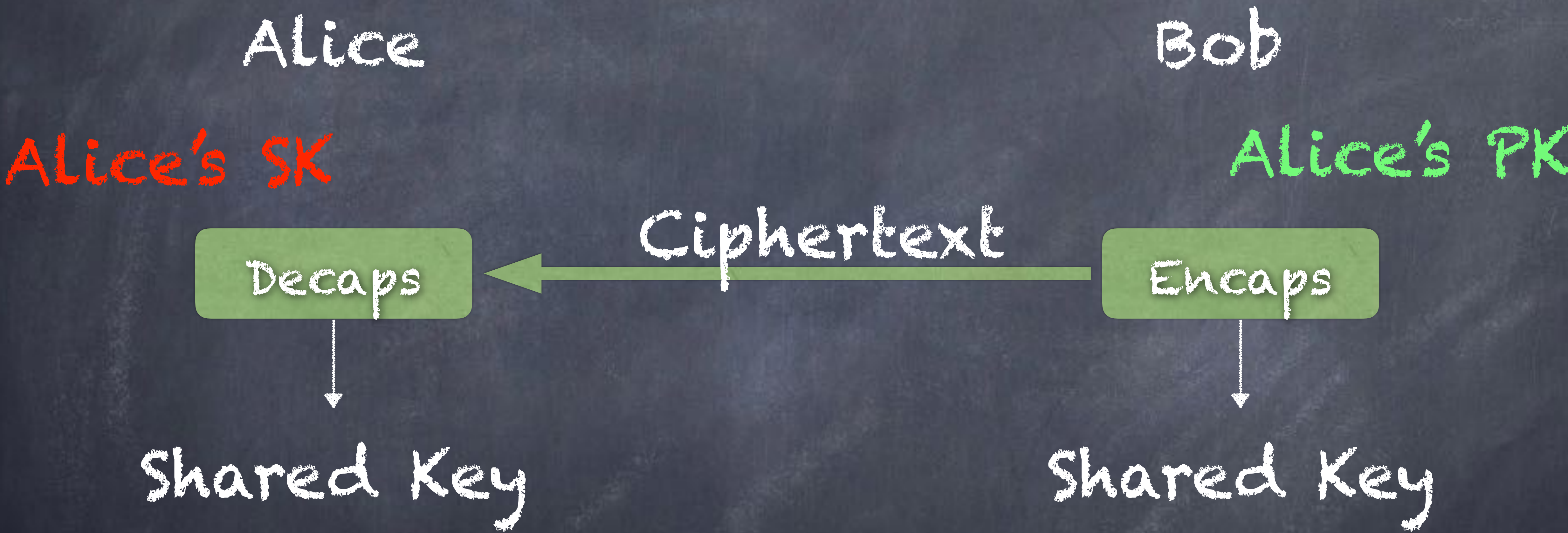
Thus Dec(Enc(m) + small error) = m

Heuristic : Error is well distributed
Allows to use simple repetition code

Multi-bit Mersenne

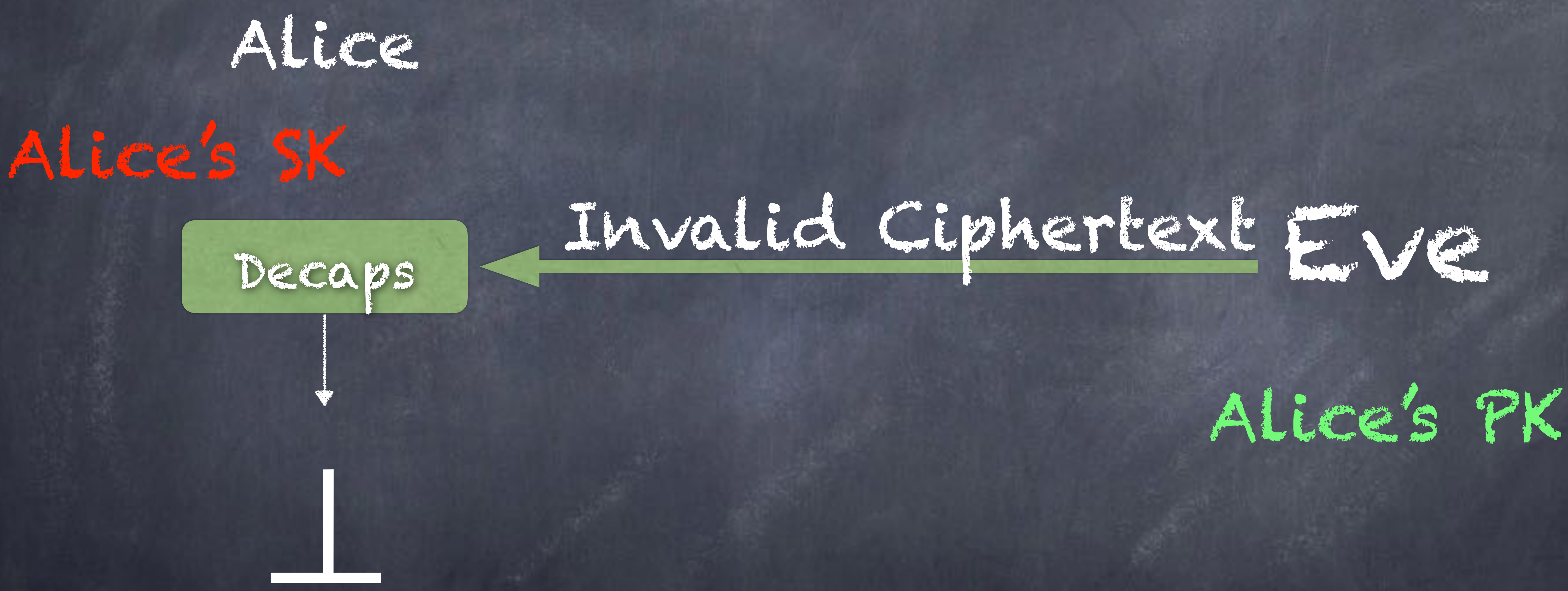
CCA-KEM

CCA-KEM



Alice
Bob
Alice's SK
Alice's PK
Ciphertext
Decaps
Encaps
Shared Key
Shared Key

CCA-KEM under active attack



Mersenne KEM encaps (with CCA security)

S = Random seed

1) Initialize PRNG/XOF from s
2) Produce pseudo random shared secret
3) Run basic encryption of s
   (getting a, b1, b2 from PRNG)
4) Output (C1, Z)

Mersenne KEM decaps (with CCA security)

1) Run basic decryption on (C1, Z)
2) Re-encapsulate from s
3) Compare and Output
    a) Shared secret
    b) or ⊥

Mersenne parameters

$n = 756839$

Low HW parameter $h=256$

Encode 256 bits:
with 2048-repetition coding

Repetition Code and Heuristic

- Window decrypts if < 1024 errors

- Only have a global bound
- Heuristic: it's well distributed

- Alternative solution:
  (Pseudo-)Randomly permute bits

Hard Problem

Distinguish

Hidden low weight

(R1, R2,
a R1+b1, a R2+b2)

a, b1, b2 with low HW

Random tuple

(R1, R2, R3, R4)

=> Semantic Security

Best Known attacks (for proposed params)

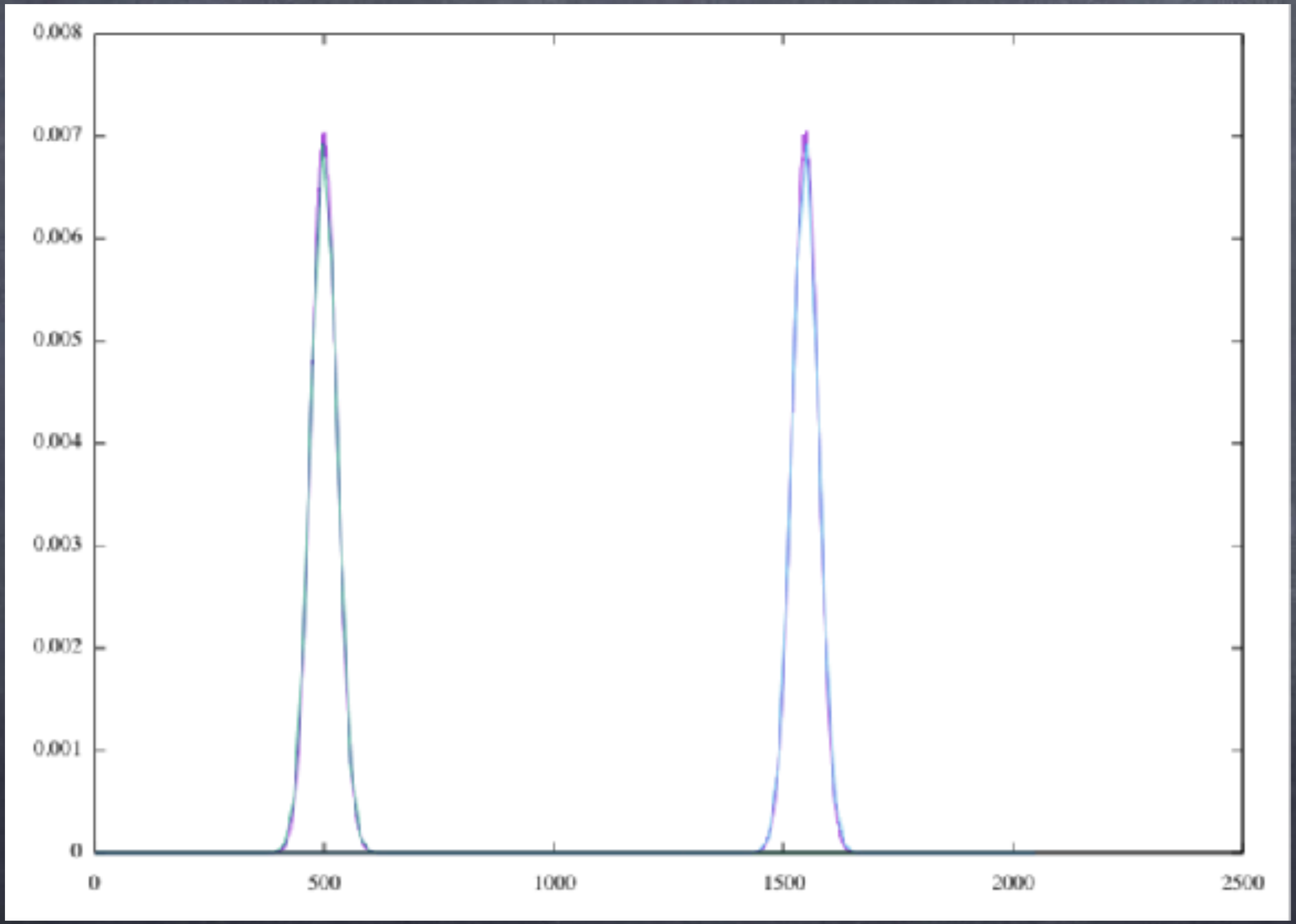
Classical : Worse than $2^{2h} \ll \binom{n-1}{h-1}^{1/2}$

Quantum : Worse than $2^{h} \ll \binom{n-1}{h-1}^{1/3}$

- (Generalized) weak key attacks
- (Noisy) birthday paradox technique

Conclusion

Heuristics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